

# EXTENSIONS OF OPERATORS, LIFTINGS OF MONADS AND DISTRIBUTIVE LAWS

LI GUO, WILLIAM KEIGHER, AND SHILONG ZHANG

**ABSTRACT.** In a previous study, the algebraic formulation of the First Fundamental Theorem of Calculus (FFTC) is shown to allow extensions of differential and Rota-Baxter operators on the one hand, and to give rise to liftings of monads and comonads, and mixed distributive laws on the other. Generalizing the FFTC, we consider in this paper a class of constraints between a differential operator and a Rota-Baxter operator. For a given constraint, we show that the existences of extensions of differential and Rota-Baxter operators, of liftings of monads and comonads, and of mixed distributive laws are equivalent. We further give a classification of the constraints satisfying these equivalent conditions.

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## 1. INTRODUCTION

An **operated algebra** is an associative algebra  $R$  together with a linear operator on  $R$ . It was introduced in 1960 by Kurosh [17]. A special case which had been studied much earlier is a **differential algebra** [19], where the linear operator  $d$  satisfies the Leibniz rule

$$(1) \quad d(xy) = d(x)y + xd(y) \text{ for all } x, y \in R.$$

More generally, for a given scalar  $\lambda$ , a **differential operator of weight  $\lambda$**  satisfies

$$(2) \quad d(xy) = d(x)y + xd(y) + \lambda d(x)d(y) \text{ for all } x, y \in R$$

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and  $d(\mathbf{1}_R) = 0$ . Another example of an operated algebra is a **Rota-Baxter algebra of weight  $\lambda$**  [2], where the operator  $P$  satisfies

$$(3) \quad P(x)P(y) = P(P(x)y) + P(xP(y)) + \lambda P(xy) \text{ for all } x, y \in R.$$

Both differential algebra and Rota-Baxter algebra arose as the algebraic abstractions of differential calculus and integral calculus, respectively. Their extensive studies have established the subjects as important areas of mathematics with broad applications in mathematics and physics [1, 4, 5, 7, 9, 13, 16, 20, 21, 25]. Bringing together the notions of a differential algebra and a Rota-Baxter algebra results in the concept of a differential Rota-Baxter algebra, where the differential operator and Rota-Baxter operator are paired through an abstraction of the First Fundamental Theorem of Calculus (FFTC). See [6, 12, 20] for a variation, called an integro-differential algebra.

As it turned out, this coupling of algebraic operators with analytic origins has important categorical implications. Indeed in [26], we gave a mixed distributive law to differential Rota-Baxter algebras. To be precise, for any algebra  $R$ , let  $(R^{\mathbb{N}}, \partial_R)$  be the cofree differential algebra on  $R$ , where  $R^{\mathbb{N}}$  denotes the Hurwitz series algebra. Let  $(\text{III}(R), P_R)$  be the free Rota-Baxter algebra on  $R$ , where  $\text{III}(R)$  is constructed by the mixable shuffle product. In [26], a differential operator on  $R$  is uniquely extended to one on  $\text{III}(R)$ , enriching  $\text{III}(R)$  to a differential Rota-Baxter algebra giving the free differential Rota-Baxter algebra. Similarly, a Rota-Baxter operator on  $R$  is uniquely extended to one on  $R^{\mathbb{N}}$ , again enriching  $R^{\mathbb{N}}$  to a differential Rota-Baxter algebra, yielding the cofree differential Rota-Baxter algebra. These extensions of operators further give the liftings of (co)monads<sup>1</sup>, which in turn give a mixed distributive law. These results suggest close connections between extensions of differential and Rota-Baxter operators, liftings of (co)monads, and mixed distributive laws.

In order to better understand the interrelationships among these properties, we should work in a broader context in which such properties can be distinguished. This is the motivation of this follow-up study. The identity in the FFTC is viewed as an instance of a polynomial identity in two noncommutative variables symbolizing the differential operator and Rota-Baxter operator, regarded as a more general constraint between the two operators exemplified by the FFTC. We explore categorical consequences of these constraints, including extensions of operators to free Rota-Baxter algebras and cofree differential algebras, liftings of (co)monads on a richer category, and existence of mixed distributive laws.

To get some sense on how things should work in general, we consider a class of constraints which is special enough to be manageable yet broad enough to include the commonly known instances and to reveal the dependence of these categorical properties on the constraints. Thus we introduce in Section 2 a class  $\Omega$  of polynomials in two noncommutative variables  $x$  and  $y$ . Each element  $\omega := \omega(x, y)$  in  $\Omega$  is regarded as a coupling of a differential operator  $d$  and a linear operator  $Q$  given by a formal identity  $\omega(d, Q) = 0$ . Then the triple  $(R, d, Q)$  will be called a type  $\omega$  operated differential algebra. Similarly, a type  $\omega$  operated Rota-Baxter algebra  $(R, q, P)$  consists of a linear operator  $q$  satisfying  $q(\mathbf{1}_R) = 0$  and a Rota-Baxter operator  $P$  with the identity  $\omega(q, P) = 0$ . As a special case, a type  $\omega$  differential Rota-Baxter algebra  $(R, d, P)$  satisfies the identity  $\omega(d, P) = 0$  between the differential operator  $d$  and Rota-Baxter operator  $P$ . The FFTC in a differential Rota-Baxter algebra corresponds to the case of  $\omega(x, y) = xy - 1$ .

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<sup>1</sup>We use the convention ‘(co)word’ to indicate the use of the notion of ‘word’ or its dual ‘coword’. This could apply to monad and extension, etc.

Let  $\mathbf{OA}$  be the category of operated algebras, and  $\mathbf{OA}_0$  be the subcategory of  $\mathbf{OA}$  which consists of all operated algebras  $(R, q)$  with the property  $q(\mathbf{1}_R) = 0$ . Let  $\mathbf{DIF}$  and  $\mathbf{RBA}$  be the categories of differential algebras and Rota-Baxter algebras, respectively. Then  $\mathbf{DIF}$  and  $\mathbf{RBA}$  are subcategories of  $\mathbf{OA}_0$  and  $\mathbf{OA}$ , respectively. Further, let  $\mathbf{ODA}_\omega$ ,  $\mathbf{ORB}_\omega$  and  $\mathbf{DRB}_\omega$  denote the categories of type  $\omega$  operated differential algebras, type  $\omega$  operated Rota-Baxter algebras and type  $\omega$  differential Rota-Baxter algebras, respectively. We provide a canonical way to extend a linear operator  $Q$  on an algebra  $R$  to one on the differential algebra  $(R^\mathbb{N}, \partial_R)$ , giving rise to a functor  $G_\omega : \mathbf{OA} \rightarrow \mathbf{ODA}_\omega$ . We likewise provide a canonical way to extend a linear operator  $q$  on an algebra  $R$  with  $q(\mathbf{1}_R) = 0$  to one on the Rota-Baxter algebra  $(\text{III}(R), P_R)$ , giving rise to a functor  $F_\omega : \mathbf{OA}_0 \rightarrow \mathbf{ORB}_\omega$ . It is natural to ask whether the restriction of  $G_\omega$  to the subcategory  $\mathbf{RBA}$  of  $\mathbf{OA}$  gives a functor  $\mathbf{RBA} \rightarrow \mathbf{DRB}_\omega$ . Likewise for  $F_\omega$ , as indicated in the following diagram.

$$\begin{array}{ccccc}
 & & \mathbf{ODA}_\omega & \longleftrightarrow & \mathbf{DRB}_\omega & \hookrightarrow & \mathbf{ORB}_\omega \\
 & \nearrow G_\omega & & \nearrow G_\omega & & \nwarrow F_\omega & \nwarrow F_\omega \\
 \mathbf{OA} & \hookrightarrow & \mathbf{RBA} & & \mathbf{DIF} & \hookrightarrow & \mathbf{OA}_0
 \end{array}$$

We show in Theorem 3.16 that this natural expectation on restrictions of functors has equivalent statements in terms of liftings of (co)monads, and corresponding mixed distributive laws. Further we provide in Theorem 4.1 a classification of those  $\omega$  that satisfy these equivalent conditions in the case that  $\mathbf{k}$  is a domain of characteristic 0.

Throughout the paper, we fix a commutative ring  $\mathbf{k}$  with identity and an element  $\lambda \in \mathbf{k}$ . Unless otherwise noted, all algebras we consider will be commutative  $\mathbf{k}$ -algebras with identity, and all operators are also  $\mathbf{k}$ -linear. All homomorphisms of algebras will be  $\mathbf{k}$ -algebra homomorphisms that preserve the identity, and all homomorphisms of operated algebras will be homomorphisms of algebras which commute with operators. Thus references to  $\mathbf{k}$  will be suppressed unless a specific  $\mathbf{k}$  is emphasized or a reminder is needed. We write  $\mathbb{N}$  for the additive monoid of natural numbers  $\{0, 1, 2, \dots\}$  and  $\mathbb{N}_+ = \{n \in \mathbb{N} \mid n > 0\}$  for the positive integers. Let  $\delta_{i,j}, i, j \in \mathbb{N}$  denote the Kronecker delta. In this paper, we use the categorical notations as in [18].

## 2. EXTENSIONS OF OPERATORS TO COFREE DIFFERENTIAL ALGEBRAS AND FREE ROTA-BAXTER ALGEBRAS

After providing background on algebras with one operator, including Rota-Baxter algebras, differential algebras and their (co)free objects, we introduce the key concepts on algebras with two operators, including type  $\omega$  operated differential algebras and type  $\omega$  operated Rota-Baxter algebras. Suitable (co)extensions of operators to these algebras are studied.

**2.1. Free Rota-Baxter algebras.** We begin with some background on Rota-Baxter algebras defined in Eq. (3). Additional details can be found in [9, 7].

Constructions of free commutative Rota-Baxter algebras on sets were first obtained by Rota and Cartier in [4, 21]. We recall from [9] the construction of the free commutative Rota-Baxter algebra  $\text{III}(A)$  of weight  $\lambda$  on a commutative algebra  $A$  with identity  $\mathbf{1}_A$ . As a module, we have

$$\text{III}(A) = \bigoplus_{i \in \mathbb{N}_+} A^{\otimes i} = A \oplus (A \otimes A) \oplus (A \otimes A \otimes A) \oplus \dots$$

where the tensors are defined over  $\mathbf{k}$ . The product for this free Rota-Baxter algebra on  $A$  is constructed in terms of a generalization of the shuffle product, called the mixable shuffle product which in its recursive form is a natural generalization of the quasi-shuffle product [14]. We will not recall details of the general construction of the mixable shuffle product in the paper. The reader is referred to [7, 9] for details and to Example 2.2 for a special case.

Define an operator  $P_A$  on  $\mathbb{III}(A)$  by assigning

$$P_A(x_0 \otimes x_1 \otimes \dots \otimes x_n) := \mathbf{1}_A \otimes x_0 \otimes x_1 \otimes \dots \otimes x_n$$

for all  $x_0 \otimes x_1 \otimes \dots \otimes x_n \in A^{\otimes(n+1)}$  and extending by additivity. We note the fact of the mixable shuffle product on  $\mathbb{III}(A)$ : for any  $u_0 \otimes u' \in A^{\otimes(n+1)}$  with  $u' \in A^{\otimes n}$ ,  $u_0 \otimes u' = u_0 P_A(u')$ .

**Theorem 2.1.** ([9, Theorem 4.1]) *The module  $\mathbb{III}(A)$ , with the mixable shuffle product, the operator  $P_A$  and the natural embedding  $j_A : A \rightarrow \mathbb{III}(A)$ , is a free Rota-Baxter algebra of weight  $\lambda$  on  $A$ . More precisely, for any Rota-Baxter algebra  $(R, P)$  of weight  $\lambda$  and any algebra homomorphism  $\varphi : A \rightarrow R$ , there exists a unique Rota-Baxter algebra homomorphism  $\tilde{\varphi} : (\mathbb{III}(A), P_A) \rightarrow (R, P)$  such that  $\varphi = \tilde{\varphi} j_A$ .*

For later applications, we give a class of Rota-Baxter algebras with non-zero Rota-Baxter operators.

**Example 2.2.** Taking  $A = \mathbf{k}$  in Theorem 2.1, then [9, Proposition 6.1] states that  $\mathbb{III}(\mathbf{k})$  is an algebra with basis  $z_i := 1^{\otimes(i+1)} \in \mathbf{k}^{\otimes(i+1)}$  for each  $i \in \mathbb{N}$ . The multiplication on  $\mathbb{III}(\mathbf{k})$  is given by

$$(4) \quad z_m z_n = \sum_{j=0}^m \binom{m+n-j}{n} \binom{n}{j} \lambda^j z_{m+n-j} \quad \text{for all } m, n \in \mathbb{N}.$$

In particular, when  $\lambda = 0$ , one sees

$$(5) \quad z_m z_n = \binom{m+n}{n} z_{m+n},$$

giving the divided power algebra.

The identity element of  $\mathbb{III}(\mathbf{k})$  is  $z_0$  and the operator  $P_{\mathbf{k}} : \mathbb{III}(\mathbf{k}) \rightarrow \mathbb{III}(\mathbf{k})$  is given by

$$P_{\mathbf{k}}(z_i) = z_{i+1} \quad \text{for each } i \in \mathbb{N}.$$

For a given  $m \in \mathbb{N}_+$ ,  $I_m := \oplus_{i \geq m} \mathbf{k} z_i$  is a Rota-Baxter ideal of  $(\mathbb{III}(\mathbf{k}), P_{\mathbf{k}})$ , that is,

$$\mathbb{III}(\mathbf{k}) I_m \subseteq I_m, \quad P_{\mathbf{k}}(I_m) \subseteq I_m,$$

giving rise to the quotient Rota-Baxter algebra  $\mathbb{III}(\mathbf{k})/I_m$ . Denoting  $\bar{u} := u + I_m \in \mathbb{III}(\mathbf{k})/I_m$  for  $u \in \mathbb{III}(\mathbf{k})$ , then  $\mathbb{III}(\mathbf{k})/I_m$  has  $\bar{z}_i, 0 \leq i \leq m-1$ , as a  $\mathbf{k}$ -basis. Its Rota-Baxter operator is

$$\bar{P}_{\mathbf{k}} : \mathbb{III}(\mathbf{k})/I_m \rightarrow \mathbb{III}(\mathbf{k})/I_m, \quad \bar{z}_i \mapsto \bar{z}_{i+1} \quad \text{for } 0 \leq i \leq m-1.$$

Consequently, when  $m \geq 2$ ,

$$\bar{P}_{\mathbf{k}}(\bar{z}_{m-2}) = \bar{z}_{m-1} \neq \bar{0}, \quad \bar{P}_{\mathbf{k}}(\bar{z}_{m-1}) = \bar{z}_m = \bar{0} \in \mathbb{III}(\mathbf{k})/I_m.$$

**2.2. Cofree differential algebras.** We review some background on differential algebras with weights, defined in Eq. (2), and refer the reader to [11] for details. Note that a differential operator of weight 0 is just a derivation in the usual sense [16].

Recall now an example that motivates the definition of a differential operator [11]. Let  $\mathbb{R}$  denote the real number field, and let  $\lambda \in \mathbb{R}$  with  $\lambda \neq 0$ . Let  $A$  denote the  $\mathbb{R}$ -algebra of  $\mathbb{R}$ -valued analytic functions on  $\mathbb{R}$ , and consider the usual ‘difference quotient’ operator  $d_\lambda$  on  $A$  defined by

$$(d_\lambda(f))(x) := \frac{f(x + \lambda) - f(x)}{\lambda} \quad \text{for all } f \in A, x \in \mathbb{R}.$$

Then  $d_\lambda$  is a differential operator of weight  $\lambda$  on  $A$ .

We next recall the concept and basic properties of the algebra of  $\lambda$ -Hurwitz series [11] as a generalization of the ring of Hurwitz series [15]. For any algebra  $A$ , let  $A^\mathbb{N}$  denote the  $\mathbf{k}$ -module of all functions  $f : \mathbb{N} \rightarrow A$ . There is a one-to-one correspondence between elements  $f \in A^\mathbb{N}$  and sequences  $(f_n) = (f_0, f_1, \dots)$  with  $f_n \in A$  given by  $f_n := f(n)$  for all  $n \in \mathbb{N}$ . The  $\lambda$ -Hurwitz product [11, § 2.3] on  $A^\mathbb{N}$  is given by

$$(6) \quad (fg)_n = \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k f_{n-j} g_{k+j} \quad \text{for all } f, g \in R^\mathbb{N}.$$

In particular, if  $\lambda = 0$ , then Eq. (6) becomes

$$(7) \quad (fg)_n = \sum_{j=0}^n \binom{n}{j} f_{n-j} g_j \quad \text{for all } f, g \in R^\mathbb{N}.$$

As in [11], we call  $A^\mathbb{N}$  the **algebra of  $\lambda$ -Hurwitz series** over  $A$ . Further, the operator

$$\partial_A : A^\mathbb{N} \rightarrow A^\mathbb{N}, \quad \partial_A(f)_n = f_{n+1} \quad \text{for all } f \in A^\mathbb{N}, n \in \mathbb{N},$$

is a differential operator of weight  $\lambda$  on  $A^\mathbb{N}$  and then  $(A^\mathbb{N}, \partial_A)$  is a differential algebra of weight  $\lambda$ . This property gives a recursive formula for  $(fg)_n$ :

$$(8) \quad (fg)_{n+1} = (\partial_A(fg))_n = (\partial_A(f)g)_n + (f\partial_A(g))_n + (\lambda\partial_A(f)\partial_A(g))_n \quad \text{for all } f, g \in A^\mathbb{N}, n \in \mathbb{N}.$$

**Proposition 2.3.** ([11, Proposition 2.8]) *For any algebra  $A$ , the differential algebra  $(A^\mathbb{N}, \partial_A)$ , together with the algebra homomorphism*

$$\varepsilon_A : A^\mathbb{N} \rightarrow A, \quad \varepsilon_A(f) := f_0 \quad \text{for all } f \in A^\mathbb{N},$$

*is a cofree differential algebra of weight  $\lambda$  on the algebra  $A$ . More precisely, for any differential algebra  $(R, d)$  of weight  $\lambda$  and any algebra homomorphism  $\varphi : R \rightarrow A$ , there exists a unique differential algebra homomorphism  $\tilde{\varphi} : (R, d) \rightarrow (A^\mathbb{N}, \partial_A)$  such that  $\varphi = \varepsilon_A \tilde{\varphi}$ .*

**Example 2.4.** On the Rota-Baxter algebra  $(\text{III}(\mathbf{k}), P_{\mathbf{k}})$  in Example 2.2, define

$$d : \text{III}(\mathbf{k}) \rightarrow \text{III}(\mathbf{k}), \quad d(z_0) = 0, d(z_n) = z_{n-1} \quad \text{for all } n \in \mathbb{N}_+.$$

Then  $(\text{III}(\mathbf{k}), d)$  is a differential algebra of weight  $\lambda$ . By ([10, Corollary 3.7]), the completion of  $(\text{III}(\mathbf{k}), d)$  is isomorphic to the algebra  $\mathbf{k}^\mathbb{N}$  of Hurwitz series over  $\mathbf{k}$ .

**2.3. (Co)extensions of operators.** For an operator on an algebra, we construct the coextension of the operator to the cofree differential algebra generated by this algebra. Further, assuming that the value of the operator on the identity of the algebra is zero, we also construct the extension of the operator to the free Rota-Baxter algebra generated by the algebra. For this purpose, we introduce a class of variations of operated algebras by enriching them with a differential operator or Rota-Baxter operator.

Let  $\mathbf{k}\langle x, y \rangle$  be the free  $\mathbf{k}$ -algebra of polynomials in two noncommutative variables  $x$  and  $y$ . Consider the subset of  $\mathbf{k}\langle x, y \rangle$ :

$$(9) \quad \Omega := xy + \mathbf{k}[x] + y\mathbf{k}[x] = \{xy - (\phi(x) + y\psi(x)) \mid \phi, \psi \in \mathbf{k}[x]\}.$$

Let  $q$  and  $Q$  be two operators on an algebra  $R$ . For each  $\omega := \omega(x, y) = xy - (\phi(x) + y\psi(x)) \in \Omega$ , we regard  $\omega(q, Q) = 0$  as a relation between  $q$  and  $Q$  which describes how the operators  $q$  and  $Q$  interact with each other. For example, when  $\omega = xy - 1$ ,  $\omega(d, P) = dP - \text{id}_R = 0$  amounts to the relation between the operators  $d$  and  $P$  in a differential Rota-Baxter algebra  $(R, d, P)$  [11] arising from the First Fundamental Theorem of Calculus.

Recall from the introduction that an operated algebra is an algebra  $R$  with a linear operator  $Q$  on  $R$ , thus denoted as a pair  $(R, Q)$ .

**Definition 2.5.** For a given  $\omega \in \Omega$  and  $\lambda \in \mathbf{k}$ , we say that the triple  $(R, d, Q)$  is a **type  $\omega$  operated differential algebra of weight  $\lambda$**  if

- (i)  $(R, d)$  is a differential algebra of weight  $\lambda$ ,
- (ii)  $(R, Q)$  is an operated algebra, and
- (iii)  $\omega(d, Q) = 0$ , that is,

$$(10) \quad dQ = \phi(d) + Q\psi(d).$$

As noted before, every  $f \in R^{\mathbb{N}}$  is identified with a sequence  $(f_n)$  of elements in  $R$ . Likewise, there is a one-to-one correspondence between operators  $\mathcal{P}$  on  $R^{\mathbb{N}}$  and sequences  $(\mathcal{P}_n)$  of linear maps where, for each  $n \in \mathbb{N}$ ,  $\mathcal{P}_n : R^{\mathbb{N}} \rightarrow R$  is given by

$$\mathcal{P}_n(f) := \mathcal{P}(f)_n \quad \text{for all } f \in R^{\mathbb{N}}.$$

For any operators  $Q, \mathcal{J}$  on  $R^{\mathbb{N}}$ , and each  $f \in R^{\mathbb{N}}, n \in \mathbb{N}$ , we obtain

$$(11) \quad (\partial_R Q)_n(f) = (\partial_R(Q(f)))_n = Q_{n+1}(f), \quad (Q\mathcal{J})_n(f) = (Q(\mathcal{J}(f)))_n = Q_n(\mathcal{J}(f)).$$

For the remainder of the paper, we prefer to use  $\mathcal{P}_n(f)$  in place of  $\mathcal{P}(f)_n$ .

We first consider coextensions of operators.

**Definition 2.6.** For a given operator  $Q : R \rightarrow R$ , we call an operator  $\widehat{Q} : R^{\mathbb{N}} \rightarrow R^{\mathbb{N}}$  a **coextension of  $Q$  to  $R^{\mathbb{N}}$**  if for all  $f \in R^{\mathbb{N}}$ , we have  $\widehat{Q}_0(f) = Q(f_0)$ .

We now establish the existence and uniqueness of a coextension.

**Proposition 2.7.** Let  $Q$  be an operator on an algebra  $R$ . For a given  $\omega = xy - (\phi(x) + y\psi(x)) \in \Omega$  with  $\phi, \psi \in \mathbf{k}[x]$ ,  $Q$  has a unique coextension  $\widehat{Q}^\omega : (R^{\mathbb{N}}, \partial_R) \rightarrow (R^{\mathbb{N}}, \partial_R)$  such that  $\omega(\partial_R, \widehat{Q}^\omega) = 0$ , that is:

$$(12) \quad \partial_R \widehat{Q}^\omega = \phi(\partial_R) + \widehat{Q}^\omega \psi(\partial_R).$$

Thus the triple  $(R^{\mathbb{N}}, \partial_R, \widehat{Q}^\omega)$  is a type  $\omega$  operated differential algebra.

*Proof.* Let  $\mathcal{P}$  be an operator on  $R^{\mathbb{N}}$ , giving by the sequence  $(\mathcal{P}_n)$  through the above one-to-one correspondence. Then applying Eq. (11), the equation  $\partial_R \mathcal{P} = \phi(\partial_R) + \mathcal{P}\psi(\partial_R)$  is equivalent to

$$(13) \quad \mathcal{P}_{n+1} = (\partial_R \mathcal{P})_n = \phi(\partial_R)_n + \mathcal{P}_n \psi(\partial_R) \quad \text{for all } n \in \mathbb{N}.$$

Thus a coextension  $\widehat{Q}$  of  $Q$  satisfying Eq. (12) is equivalent to a solution  $(\widehat{Q}_n)$  of the recursion in Eq. (13) with  $\mathcal{P} = \widehat{Q}$  and with the initial condition

$$(14) \quad \widehat{Q}_0(f) = Q(f_0) \quad \text{for all } f \in R^{\mathbb{N}}.$$

Then the proposition follows since this recursion has a unique solution.  $\square$

**Proposition 2.8.** *Let  $(R, Q)$  be an operated algebra and  $\widehat{Q}$  be a coextension of  $Q$  to  $R^{\mathbb{N}}$ .*

(i)  *$(R, Q)$  is a Rota-Baxter algebra of weight  $\lambda$  if and only if*

$$(15) \quad \widehat{Q}_0(f)\widehat{Q}_0(g) = \widehat{Q}_0(\widehat{Q}(f)g) + \widehat{Q}_0(f\widehat{Q}(g)) + \lambda\widehat{Q}_0(fg) \quad \text{for all } f, g \in R^{\mathbb{N}}.$$

(ii)  *$(R^{\mathbb{N}}, \widehat{Q})$  is a Rota-Baxter algebra of weight  $\lambda$  if and only if for all  $f, g \in R^{\mathbb{N}}$ ,  $n \in \mathbb{N}$ ,*

$$(16) \quad \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \widehat{Q}_{n-j}(f) \widehat{Q}_{k+j}(g) = \widehat{Q}_n(\widehat{Q}(f)g) + \widehat{Q}_n(f\widehat{Q}(g)) + \lambda\widehat{Q}_n(fg).$$

*Proof.* (i). This follows since the left hand side of Eq. (15) is  $Q(f_0)Q(g_0)$  while, by Eq. (6), the right hand side of Eq. (15) is

$$Q((Q(f)g)_0) + Q((fQ(g))_0) + \lambda Q((fg)_0) = Q(Q(f_0)g_0) + Q(f_0Q(g_0)) + \lambda Q(f_0g_0) \quad \text{for all } f, g \in R^{\mathbb{N}}.$$

(ii). The coextension  $\widehat{Q}$  is a Rota-Baxter operator of weight  $\lambda$  if and only if

$$\widehat{Q}(f)\widehat{Q}(g) = \widehat{Q}(\widehat{Q}(f)g) + \widehat{Q}(f\widehat{Q}(g)) + \lambda\widehat{Q}(fg) \quad \text{for all } f, g \in R^{\mathbb{N}}.$$

Applying Eq. (6), this means

$$\sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \widehat{Q}_{n-j}(f) \widehat{Q}_{k+j}(g) = \widehat{Q}_n(\widehat{Q}(f)g) + \widehat{Q}_n(f\widehat{Q}(g)) + \lambda\widehat{Q}_n(fg) \quad \text{for all } n \in \mathbb{N}.$$

$\square$

Next, we consider extensions of operators.

**Definition 2.9.** *For a given  $\omega \in \Omega$  and  $\lambda \in \mathbf{k}$ , we say that the triple  $(R, q, P)$  is a **type  $\omega$  operated Rota-Baxter algebra of weight  $\lambda$**  if*

- (i)  *$(R, q)$  is an operated algebra with the property  $q(\mathbf{1}_R) = 0$ ,*
- (ii)  *$(R, P)$  is a Rota-Baxter algebra of weight  $\lambda$ , and*
- (iii)  *$\omega(q, P) = 0$ , that is,*

$$qP = \phi(q) + P\psi(q).$$

**Definition 2.10.** For a given operator  $q : R \rightarrow R$  satisfying  $q(\mathbf{1}_R) = 0$ , we call an operator  $\hat{q} : \text{III}(R) \rightarrow \text{III}(R)$  an **extension** of  $q$  to  $\text{III}(R)$  if  $\hat{q}|_R = q$ , that is,  $\hat{q}_1 = q$ .

By the definition of the mixable shuffle product on  $\text{III}(R)$ , we obtain  $\mathbf{1}_{\text{III}(R)} = \mathbf{1}_R$ . Then an extension  $\hat{q}$  of  $q$  to  $\text{III}(R)$  satisfies  $\hat{q}(\mathbf{1}_{\text{III}(R)}) = q(\mathbf{1}_R) = 0$ .



**Proposition 2.11.** *Let  $(R, q)$  be an operated algebra where the operator  $q$  satisfies  $q(\mathbf{1}_R) = 0$ . For a given  $\omega = xy - (\phi(x) + y\psi(x)) \in \Omega$  with  $\phi, \psi \in \mathbf{k}[x]$ ,  $q$  has a unique extension  $\hat{q}^\omega : (\text{III}(R), P_R) \rightarrow (\text{III}(R), P_R)$  with the following property: for  $u = u_0 \otimes u' \in R^{\otimes(n+1)}$  with  $u' \in R^{\otimes n}$ ,*

$$(17) \quad \hat{q}^\omega(u) = q(u_0) \otimes u' + (u_0 + \lambda q(u_0))(\phi(\hat{q}^\omega) + P_R \psi(\hat{q}^\omega))(u')$$

and

$$(18) \quad \hat{q}^\omega(\oplus_{i=1}^n R^{\otimes i}) \subseteq \oplus_{i=1}^n R^{\otimes i} \quad \text{for each } n \in \mathbb{N}_+.$$

Note that Eq. (17) implies  $\hat{q}^\omega P_R = \phi(\hat{q}^\omega) + P_R \psi(\hat{q}^\omega)$ . Thus the triple  $(\text{III}(R), \hat{q}^\omega, P_R)$  is a type  $\omega$  operated Rota-Baxter algebra.

*Proof.* Since  $\text{III}(R)$  is the direct limit (via taking union) of its submodules  $\oplus_{i=1}^n R^{\otimes i}$ ,  $n \in \mathbb{N}_+$ , by [22, Proposition 5.26], there is a one-to-one correspondence between operators  $\mathcal{D} : \text{III}(R) \rightarrow \text{III}(R)$  and compatible sequences  $(\mathcal{D}_n)$  of linear maps  $\mathcal{D}_n : \oplus_{i=1}^n R^{\otimes i} \rightarrow \text{III}(R)$  where the compatible condition means  $\mathcal{D}_n = \mathcal{D}_{n+1}|_{\oplus_{i=1}^n R^{\otimes i}}$  for each  $n \in \mathbb{N}_+$ .

Thus we just need to show by induction on  $n \in \mathbb{N}_+$  that there is a unique compatible sequence  $(\hat{q}_n^\omega)$  with  $\hat{q}_n^\omega : \oplus_{i=1}^n R^{\otimes i} \rightarrow \text{III}(R)$  extending  $q$  and satisfying Eqs. (17), (18) when  $\hat{q}^\omega$  is replaced by  $\hat{q}_n^\omega$ .

As  $\hat{q}^\omega : \text{III}(R) \rightarrow \text{III}(R)$  is an extension of  $q$ , we have  $\hat{q}_1^\omega = q$  and  $\hat{q}_1^\omega(R) \subseteq R$ , verifying the case when  $n = 1$ .

For a given  $k \in \mathbb{N}_+$ , assume that the required  $\hat{q}_k^\omega$  on  $\oplus_{i=1}^k R^{\otimes i}$  has been defined. Define  $\hat{q}_{k+1}^\omega$  on  $\oplus_{i=1}^{k+1} R^{\otimes i}$  by first taking  $\hat{q}_{k+1}^\omega|_{\oplus_{i=1}^k R^{\otimes i}} = \hat{q}_k^\omega$ . Further consider  $v \in R^{\otimes(k+1)}$  and write  $v = v_0 \otimes v'$  with  $v' \in R^{\otimes k}$ . By the induction hypothesis,  $(\phi(\hat{q}_{k+1}^\omega))(v') = (\phi(\hat{q}_k^\omega))(v')$  and  $(\psi(\hat{q}_{k+1}^\omega))(v') = (\psi(\hat{q}_k^\omega))(v')$  are also defined and satisfy

$$(\phi(\hat{q}_{k+1}^\omega))(v') \subseteq \oplus_{i=1}^k R^{\otimes i}, \quad (\psi(\hat{q}_{k+1}^\omega))(v') \subseteq \oplus_{i=1}^k R^{\otimes i}.$$

Then we can uniquely define

$$\hat{q}_{k+1}^\omega(v) := q(v_0) \otimes v' + (v_0 + \lambda q(v_0))(\phi(\hat{q}_{k+1}^\omega) + P_R \psi(\hat{q}_{k+1}^\omega))(v').$$

Then  $\hat{q}_{k+1}^\omega(v) \in \oplus_{i=1}^{k+1} R^{\otimes i}$ . This completes the construction of the desired  $\hat{q}_{k+1}^\omega$  and the induction.  $\square$

### 3. LIFTINGS OF COMONADS AND MONADS, AND MIXED DISTRIBUTIVE LAWS

In this section, we characterize (co)extensions of operators in terms of liftings of (co)monads, as well as mixed distributive laws.

**3.1. The monad giving Rota-Baxter algebras and comonad giving differential algebras.** We first recall the monad giving Rota-Baxter algebras.

We let **ALG** denote the category of commutative algebras, and let **RBA** $_\lambda$ , or simply **RBA**, denote the category of commutative Rota-Baxter algebras of weight  $\lambda$ .

Let  $U : \mathbf{RBA} \rightarrow \mathbf{ALG}$  denote the forgetful functor by forgetting the Rota-Baxter operator. Let  $F : \mathbf{ALG} \rightarrow \mathbf{RBA}$  denote the functor given on objects  $A$  in **ALG** by  $F(A) = (\text{III}(A), P_A)$  and on morphisms  $\varphi : A \rightarrow B$  in **ALG** by

$$F(\varphi) \left( \sum_{i=1}^k a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \right) = \sum_{i=1}^k \varphi(a_{i0}) \otimes \varphi(a_{i1}) \otimes \cdots \otimes \varphi(a_{in_i})$$



for any  $\sum_{i=1}^k a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \in \text{III}(A)$ .

Define a natural transformation

$$\eta : \text{id}_{\mathbf{ALG}} \rightarrow UF$$

with

$$\eta_R : R \rightarrow (UF)(R) = \text{III}(R) \quad \text{for any } R \in \mathbf{ALG}$$

to be just the natural embedding  $j_R : R \rightarrow \text{III}(R)$ . There is also a natural transformation

$$\varepsilon : FU \rightarrow \text{id}_{\mathbf{RBA}}$$

with

$$\varepsilon_{(R,P)} : (FU)(R, P) = (\text{III}(R), P_R) \rightarrow (R, P) \quad \text{for each } (R, P) \in \mathbf{RBA},$$

defined by

$$(19) \quad \varepsilon_{(R,P)} \left( \sum_{i=1}^k u_{i0} \otimes u_{i1} \otimes \cdots \otimes u_{in_i} \right) = \sum_{i=1}^k u_{i0} P(u_{i1} P(\cdots P(u_{in_i}) \cdots))$$

for any  $\sum_{i=1}^k u_{i0} \otimes u_{i1} \otimes \cdots \otimes u_{in_i} \in \text{III}(R)$ .

From a general principle of category theory [18, 26], equivalent to Theorem 2.1, we have

**Corollary 3.1.** *The functor  $F : \mathbf{ALG} \rightarrow \mathbf{RBA}$  defined above is the left adjoint of the forgetful functor  $U : \mathbf{RBA} \rightarrow \mathbf{ALG}$ . In other words, there is an adjunction  $\langle F, U, \eta, \varepsilon \rangle : \mathbf{ALG} \rightarrow \mathbf{RBA}$ .*

The adjunction  $\langle F, U, \eta, \varepsilon \rangle : \mathbf{ALG} \rightarrow \mathbf{RBA}$  gives rise to a monad  $\mathbf{T} = \mathbf{T}_{\mathbf{RBA}} = \langle T, \eta, \mu \rangle$  on  $\mathbf{ALG}$ , where  $T$  is the functor  $T := UF : \mathbf{ALG} \rightarrow \mathbf{ALG}$  and  $\mu$  is the natural transformation defined by  $\mu := U\varepsilon F : TT \rightarrow T$ .

From [18], the monad  $\mathbf{T}$  induces a category of  $\mathbf{T}$ -algebras, denoted by  $\mathbf{ALG}^{\mathbf{T}}$ . The objects in  $\mathbf{ALG}^{\mathbf{T}}$  are pairs  $\langle A, h \rangle$ , where  $A$  is in  $\mathbf{ALG}$  and  $h : \text{III}(A) \rightarrow A$  is an algebra homomorphism satisfying

$$h\eta_A = \text{id}_A, \quad hT(h) = h\mu_A.$$

A morphism  $\phi : \langle R, f \rangle \rightarrow \langle S, g \rangle$  in  $\mathbf{ALG}^{\mathbf{T}}$  is an algebra homomorphism  $\phi : R \rightarrow S$  such that  $gT(\phi) = \phi f$ .

Further, the monad  $\mathbf{T}$  gives rise to an adjunction

$$\langle F^{\mathbf{T}}, U^{\mathbf{T}}, \eta^{\mathbf{T}}, \varepsilon^{\mathbf{T}} \rangle : \mathbf{ALG} \rightarrow \mathbf{ALG}^{\mathbf{T}},$$

where the functor

$$F^{\mathbf{T}} : \mathbf{ALG} \rightarrow \mathbf{ALG}^{\mathbf{T}}$$

is defined on objects  $A$  in  $\mathbf{ALG}$  by  $F^{\mathbf{T}}(A) = \langle \text{III}(A), \mu_A \rangle$ . The functor

$$U^{\mathbf{T}} : \mathbf{ALG}^{\mathbf{T}} \rightarrow \mathbf{ALG}$$

is given on objects  $\langle A, h \rangle$  in  $\mathbf{ALG}^{\mathbf{T}}$  by  $U^{\mathbf{T}}\langle A, h \rangle = A$ . The natural transformations  $\eta^{\mathbf{T}}$  and  $\varepsilon^{\mathbf{T}}$  are defined similarly as  $\eta$  and  $\varepsilon$ , respectively. Then there is a uniquely defined comparison functor

$$K : \mathbf{RBA} \rightarrow \mathbf{ALG}^{\mathbf{T}}, \quad K(R, P) = \langle R, U(\varepsilon_{(R,P)}) \rangle \quad \text{for any } (R, P) \in \mathbf{RBA}$$

such that  $KF = F^{\mathbf{T}}$  and  $U^{\mathbf{T}}K = U$ .

**Corollary 3.2.** ([26, Corollary 2.6]) *The comparison functor  $K : \mathbf{RBA} \rightarrow \mathbf{ALG}^{\mathbf{T}}$  is an isomorphism, that is,  $\mathbf{RBA}$  is monadic over  $\mathbf{ALG}$ .*

Next, we recall the comonad giving differential algebras.

Let  $\varphi : A \rightarrow B$  be an algebra homomorphism. Then the map

$$(20) \quad \varphi^{\mathbb{N}} : A^{\mathbb{N}} \rightarrow B^{\mathbb{N}}, \quad \varphi_n^{\mathbb{N}}(f) := \varphi(f_n) \quad \text{for all } f \in A^{\mathbb{N}}, n \in \mathbb{N},$$

is a differential algebra homomorphism from  $(A^{\mathbb{N}}, \partial_A)$  to  $(B^{\mathbb{N}}, \partial_B)$ .

Let **DIF** denote the category of differential algebras of weight  $\lambda$ . Let  $V : \mathbf{DIF} \rightarrow \mathbf{ALG}$  denote the forgetful functor. We also have a functor  $G : \mathbf{ALG} \rightarrow \mathbf{DIF}$  given on objects  $A$  in **ALG** by  $G(A) := (A^{\mathbb{N}}, \partial_A)$  and on morphisms  $\varphi : A \rightarrow B$  in **ALG** by  $G(\varphi) := \varphi^{\mathbb{N}}$  as defined in Eq. (20).

There is a natural transformation  $\eta : \text{id}_{\mathbf{DIF}} \rightarrow GV$  by

$$(21) \quad \eta_{(R,d)} : (R, d) \rightarrow (GV)(R, d) = (R^{\mathbb{N}}, \partial_R), \quad (\eta_{(R,d)}(x))_n := d^n(x)$$

for any  $(R, d) \in \mathbf{DIF}$  and all  $x \in R, n \in \mathbb{N}$ . There is also a natural transformation  $\varepsilon : VG \rightarrow \text{id}_{\mathbf{ALG}}$  by

$$(22) \quad \varepsilon_A : (VG)(A) = A^{\mathbb{N}} \rightarrow A, \quad \varepsilon_A(f) := f_0 \quad \text{for any } A \in \mathbf{ALG}, f \in A^{\mathbb{N}}.$$

As an equivalent statement of Proposition 2.3, we have

**Corollary 3.3.** *The functor  $G : \mathbf{ALG} \rightarrow \mathbf{DIF}$  is the right adjoint of the forgetful functor  $V : \mathbf{DIF} \rightarrow \mathbf{ALG}$ . In other words, there is an adjunction  $\langle V, G, \eta, \varepsilon \rangle : \mathbf{DIF} \rightarrow \mathbf{ALG}$ .*

Corresponding to the adjunction  $\langle V, G, \eta, \varepsilon \rangle : \mathbf{DIF} \rightarrow \mathbf{ALG}$ , there is a comonad  $\mathbf{C} = \langle C, \varepsilon, \delta \rangle$  on **ALG**, where  $C$  is the functor  $C := VG : \mathbf{ALG} \rightarrow \mathbf{ALG}$  and  $\delta$  is the natural transformation from  $C$  to  $CC$  defined by  $\delta := V\eta G$ .

The comonad  $\mathbf{C}$  induces a category of  $\mathbf{C}$ -coalgebras, denoted by  $\mathbf{ALG}_{\mathbf{C}}$ . The objects in  $\mathbf{ALG}_{\mathbf{C}}$  are pairs  $\langle A, f \rangle$ , where  $A$  is in **ALG** and  $f : A \rightarrow A^{\mathbb{N}}$  is a homomorphism in **ALG** satisfying the properties

$$\varepsilon_A f = \text{id}_A, \quad \delta_A f = f^{\mathbb{N}} f.$$

A morphism  $\varphi : \langle A, f \rangle \rightarrow \langle B, g \rangle$  in  $\mathbf{ALG}_{\mathbf{C}}$  is an algebra homomorphism  $\varphi : A \rightarrow B$  such that  $g\varphi = \varphi^{\mathbb{N}} f$ .

The comonad  $\mathbf{C}$  also gives rise to an adjunction

$$\langle V_{\mathbf{C}}, G_{\mathbf{C}}, \eta_{\mathbf{C}}, \varepsilon_{\mathbf{C}} \rangle : \mathbf{ALG}_{\mathbf{C}} \rightarrow \mathbf{ALG},$$

where

$$V_{\mathbf{C}} : \mathbf{ALG}_{\mathbf{C}} \rightarrow \mathbf{ALG}$$

is given on objects  $\langle R, f \rangle$  in  $\mathbf{ALG}_{\mathbf{C}}$  by  $V_{\mathbf{C}}\langle R, f \rangle = R$ . The functor

$$G_{\mathbf{C}} : \mathbf{ALG} \rightarrow \mathbf{ALG}_{\mathbf{C}}$$

is defined on objects  $A$  in **ALG** by  $G_{\mathbf{C}}(A) = \langle A^{\mathbb{N}}, \delta_A \rangle$ . The natural transformations  $\eta_{\mathbf{C}}$  and  $\varepsilon_{\mathbf{C}}$  are defined similarly to  $\eta$  and  $\varepsilon$  in Eqs. (21) and (22), respectively. Consequently there is a uniquely defined cocomparison functor  $H : \mathbf{DIF} \rightarrow \mathbf{ALG}_{\mathbf{C}}$  such that  $HG = G_{\mathbf{C}}$  and  $V_{\mathbf{C}}H = V$ .

**Corollary 3.4.** ([26, Corollary 3.5]) *The cocomparison functor  $H : \mathbf{DIF} \rightarrow \mathbf{ALG}_{\mathbf{C}}$  is an isomorphism, i.e., **DIF** is comonadic over **ALG**.*

**3.2. Lifting comonads on RBA.** In view of the categorical study, we rephrase Proposition 2.7 in terms of categories of various operated algebras. See [8, 17] for related studies.

Recall from Eq. (9) that

$$\Omega := xy + \mathbf{k}[x] + y\mathbf{k}[x] = \{xy - (\phi(x) + y\psi(x)) \mid \phi, \psi \in \mathbf{k}[x]\}.$$

Let  $\mathbf{OA}$  denote the category of operated algebras and, for a given  $\omega \in \Omega$ , let  $\mathbf{ODA}_\omega$  denote the category of type  $\omega$  operated differential algebras of weight  $\lambda$  in Definition 2.5. Thanks to Proposition 2.7, we obtain a functor

$$(23) \quad G_\omega : \mathbf{OA} \rightarrow \mathbf{ODA}_\omega$$

given on objects  $(R, Q)$  in  $\mathbf{OA}$  by  $G_\omega(R, Q) = (R^\mathbb{N}, \partial_R, \widehat{Q}^\omega)$  and on morphisms  $\varphi : (R, Q) \rightarrow (S, P)$  in  $\mathbf{OA}$  by  $(G_\omega(\varphi))_n(f) := \varphi(f_n)$  for any  $f \in R^\mathbb{N}$  and  $n \in \mathbb{N}$ .

**Definition 3.5.** For a given  $\omega \in \Omega$  and  $\lambda \in \mathbf{k}$ , we say that the triple  $(R, d, P)$  is a **type  $\omega$  differential Rota-Baxter algebra of weight  $\lambda$**  if

- (i)  $(R, d)$  is a differential algebra of weight  $\lambda$ ,
- (ii)  $(R, P)$  is a Rota-Baxter algebra of weight  $\lambda$ , and
- (iii)  $\omega(d, P) = 0$ , that is,

$$(24) \quad dP = \phi(d) + P\psi(d).$$

The category of type  $\omega$  differential Rota-Baxter algebras of weight  $\lambda$  will be denoted by  $\mathbf{DRB}_\omega$ . Note that  $\mathbf{DRB}_\omega$  is a subcategory of  $\mathbf{ODA}_\omega$ .

We make the following assumption for a given  $\omega \in \Omega$  for the rest of Section 3.2:

**Assumption 3.6.** For every Rota-Baxter algebra  $(R, P)$  of weight  $\lambda$ , the coextension  $\widehat{P}^\omega$  of  $P$  in Proposition 2.7 also gives a Rota-Baxter algebra  $(R^\mathbb{N}, \widehat{P}^\omega)$  of weight  $\lambda$ .

This assumption amounts to the requirement that the functor  $G_\omega : \mathbf{OA} \rightarrow \mathbf{ODA}_\omega$  in Eq. (23) restricts to a functor

$$G_\omega : \mathbf{RBA} \rightarrow \mathbf{DRB}_\omega.$$

Let  $V_\omega : \mathbf{ODA}_\omega \rightarrow \mathbf{OA}$  denote the forgetful functor by forgetting the differential structure. Then  $V_\omega$  restricts to a functor

$$V_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}.$$

**Lemma 3.7.** Let  $\omega \in \Omega$  satisfy Assumption 3.6, and  $(R, d, P) \in \mathbf{DRB}_\omega$ . Then we have

$$(25) \quad \widehat{P}^\omega \eta_{(R,d)} = \eta_{(R,d)} P,$$

where  $\eta_{(R,d)}$  is given in Eq. (21).

*Proof.* For all  $n \in \mathbb{N}$ , we obtain  $(\widehat{P}^\omega \eta_{(R,d)})_n = \widehat{P}_n^\omega \eta_{(R,d)}$  and  $(\eta_{(R,d)} P)_n = d^n P$ . Then Eq. (25) is equivalent to

$$(26) \quad \widehat{P}_n^\omega \eta_{(R,d)} = d^n P \quad \text{for all } n \in \mathbb{N}.$$

We will prove by induction on  $n \in \mathbb{N}$  that Eq. (26) holds.

First for any  $u \in R$ ,

$$(\widehat{P}_0^\omega \eta_{(R,d)})(u) = P(\eta_{(R,d)}(u)_0) = P(u) = (d^0 P)(u).$$

Next assume that for a given  $k \in \mathbb{N}$ ,

$$(27) \quad \widehat{P}_k^\omega \eta_{(R,d)} = d^k P$$

holds. For every  $v \in R$ , we have

$$((\partial_R^i)_\ell \eta_{(R,d)})(v) = d^{\ell+i}(v) = d^\ell(d^i(v)) \quad \text{for all } i, \ell \in \mathbb{N}.$$

Then

$$(28) \quad \phi(\partial_R)_k \eta_{(R,d)} = d^k \phi(d), \quad \psi(\partial_R) \eta_{(R,d)} = \eta_{(R,d)} \psi(d).$$

Since

$$\begin{aligned} (\widehat{P}_{k+1}^\omega \eta_{(R,d)})(v) &= \widehat{P}_{k+1}^\omega(\eta_{(R,d)}(v)) \\ &= (\phi(\partial_R)_k + \widehat{P}_k^\omega \psi(\partial_R))(\eta_{(R,d)}(v)) \quad (\text{by Eq. (13)}) \\ &= (d^k \phi(d))(v) + ((\widehat{P}_k^\omega \eta_{(R,d)}) \psi(d))(v) \quad (\text{by Eq. (28)}) \\ &= (d^k \phi(d))(v) + ((d^k P) \psi(d))(v) \quad (\text{by Eq. (27)}) \end{aligned}$$

and

$$\begin{aligned} (d^{k+1} P)(v) &= (d^k(dP))(v) \\ &= (d^k(\phi(d) + P\psi(d)))(v) \quad (\text{by Eq. (10)}) \\ &= (d^k \phi(d))(v) + (d^k(P\psi(d)))(v), \end{aligned}$$

we obtain  $(\widehat{P}_{k+1}^\omega \eta_{(R,d)})(v) = (d^{k+1} P)(v)$ . This completes the induction.  $\square$

**Proposition 3.8.** *Let  $\omega \in \Omega$  satisfy Assumption 3.6. The functor  $G_\omega : \mathbf{RBA} \rightarrow \mathbf{DRB}_\omega$  is the right adjoint of the forgetful functor  $V_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}$ . In other words, there is an adjunction  $\langle V_\omega, G_\omega, \eta^\omega, \varepsilon^\omega \rangle : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}$ .*

*Proof.* By [18], it is equivalent to show that there are two natural transformations  $\varepsilon^\omega : V_\omega G_\omega \rightarrow \text{id}_{\mathbf{RBA}}$  and  $\eta^\omega : \text{id}_{\mathbf{DRB}_\omega} \rightarrow G_\omega V_\omega$  that satisfy the equations

$$G_\omega \varepsilon^\omega \circ \eta^\omega G_\omega = G_\omega, \quad \varepsilon^\omega V_\omega \circ V_\omega \eta^\omega = V_\omega.$$

For any  $(A, P) \in \mathbf{RBA}$ , define

$$\varepsilon_{(A,P)}^\omega : (V_\omega G_\omega)(A, P) = (A^\mathbb{N}, \widehat{P}^\omega) \rightarrow (A, P), \quad \varepsilon_{(A,P)}^\omega(f) := f_0 = \varepsilon_A(f) \quad \text{for all } f \in A^\mathbb{N}.$$

By Proposition 2.3,  $\varepsilon_{(A,P)}^\omega$  is an algebra homomorphism. By

$$\varepsilon_{(A,P)}^\omega(\widehat{P}^\omega(f)) = P(f_0) = P(\varepsilon_{(A,P)}^\omega(f)),$$

we find that  $\varepsilon_{(A,P)}^\omega$  is a homomorphism of Rota-Baxter algebras. If  $\varphi : (A, P) \rightarrow (A', P')$  is any homomorphism of Rota-Baxter algebras, then

$$\varepsilon_{(A',P')}^\omega(V_\omega G_\omega)(\varphi) = \varphi \varepsilon_{(A,P)}^\omega,$$

that is,  $\varepsilon^\omega$  is a natural transformation.

For any  $(R, d, P) \in \mathbf{DRB}_\omega$ , define  $\eta_{(R,d,P)}^\omega : (R, d, P) \rightarrow (R^\mathbb{N}, \partial_R, \widehat{P}^\omega)$  by

$$(\eta_{(R,d,P)}^\omega(x))_n := d^n(x) = (\eta_{(R,d)}(x))_n$$

for all  $x \in R$  and  $n \in \mathbb{N}$ . Then  $\eta_{(R,d,P)}^\omega$  is an algebra homomorphism. Also we obtain

$$\partial_R \eta_{(R,d,P)}^\omega = \eta_{(R,d,P)}^\omega d$$

and  $\widehat{P}^\omega \eta_{(R,d,P)}^\omega = \eta_{(R,d,P)}^\omega P$  follows from Lemma 3.7 and  $\eta_{(R,d,P)}^\omega = \eta_{(R,d)}$ . Then  $\eta_{(R,d,P)}^\omega$  is a morphism in  $\mathbf{DRB}_\omega$ . Furthermore, if  $\varphi : (R, d, P) \rightarrow (R', d', P')$  is a morphism in  $\mathbf{DRB}_\omega$ , then one sees that  $\eta_{(R',d',P')}^\omega \varphi = (G_\omega V_\omega)(\varphi) \eta_{(R,d,P)}^\omega$ . Hence  $\eta^\omega$  is a natural transformation.

To check that  $G_\omega \varepsilon^\omega \circ \eta^\omega G_\omega = G_\omega$ , let  $(A, P) \in \mathbf{RBA}$ ,  $f \in A^\mathbb{N}$  and  $n \in \mathbb{N}$ . Then

$$(G_\omega \varepsilon_{(A,P)}^\omega (\eta_{(A^\mathbb{N}, \partial_A, \widehat{P}^\omega)}^\omega (f)))_n = \varepsilon_{(A,P)}^\omega (\eta_{(A^\mathbb{N}, \partial_A, \widehat{P}^\omega)}^\omega (f)_n) = \varepsilon_{(A,P)}^\omega (\partial_A^n (f)) = (\partial_A^n (f))_0 = f_{0+n} = f_n.$$

Similarly, to check that  $\varepsilon^\omega V_\omega \circ V_\omega \eta^\omega = V_\omega$ , let  $(R, d, P) \in \mathbf{DRB}_\omega$  and  $x \in R$ . Then

$$\varepsilon_{(R,P)}^\omega (\eta_{(R,d,P)}^\omega (x)) = (\eta_{(R,d,P)}^\omega (x))_0 = d^0(x) = x,$$

as desired.  $\square$

The adjunction  $\langle V_\omega, G_\omega, \eta^\omega, \varepsilon^\omega \rangle : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}$  gives a comonad  $\mathbf{C}_\omega = \langle C_\omega, \varepsilon^\omega, \delta^\omega \rangle$  on the category  $\mathbf{RBA}$ , where  $C_\omega : \mathbf{RBA} \rightarrow \mathbf{RBA}$  is the functor whose value for any  $(R, P) \in \mathbf{RBA}$  is  $C_\omega(R, P) = (R^\mathbb{N}, \widehat{P}^\omega)$  and  $\delta^\omega$  is a natural transformation from  $C_\omega$  to  $C_\omega C_\omega$  defined by  $\delta^\omega := V_\omega \eta^\omega G_\omega$ . In other words, for any  $(R, P) \in \mathbf{RBA}$ ,

$$\delta_{(R,P)}^\omega : (R^\mathbb{N}, \widehat{P}^\omega) \rightarrow ((R^\mathbb{N})^\mathbb{N}, \widehat{\widehat{P}^\omega}^\omega), \quad \delta_{(R,P)}^\omega(f) = \delta_R(f) \quad \text{for all } f \in R^\mathbb{N}.$$

Consequently, the comonad  $\mathbf{C}_\omega$  is a lifting of  $\mathbf{C}$  on  $\mathbf{RBA}$ .

Similarly, there is a category of  $\mathbf{C}_\omega$ -coalgebras, denoted by  $\mathbf{RBA}_{\mathbf{C}_\omega}$ . The comonad  $\mathbf{C}_\omega$  also induces an adjunction

$$\langle (V_\omega)_{\mathbf{C}_\omega}, (G_\omega)_{\mathbf{C}_\omega}, \eta_{\mathbf{C}_\omega}^\omega, \varepsilon_{\mathbf{C}_\omega}^\omega \rangle : \mathbf{RBA}_{\mathbf{C}_\omega} \rightarrow \mathbf{RBA},$$

where

$$(V_\omega)_{\mathbf{C}_\omega} : \mathbf{RBA}_{\mathbf{C}_\omega} \rightarrow \mathbf{RBA}$$

is given on objects  $\langle (A, P), f \rangle$  in  $\mathbf{RBA}_{\mathbf{C}_\omega}$  by  $(V_\omega)_{\mathbf{C}_\omega} \langle (A, P), f \rangle = (A, P)$  and on morphisms  $\varphi : \langle (A, P), f \rangle \rightarrow \langle (B, Q), g \rangle$  in  $\mathbf{RBA}_{\mathbf{C}_\omega}$  by  $(V_\omega)_{\mathbf{C}_\omega}(\varphi) = \varphi$ . The functor

$$(G_\omega)_{\mathbf{C}_\omega} : \mathbf{RBA} \rightarrow \mathbf{RBA}_{\mathbf{C}_\omega}$$

is defined on objects  $(A, P)$  in  $\mathbf{RBA}$  by  $(G_\omega)_{\mathbf{C}_\omega}(A, P) = \langle (A^\mathbb{N}, \widehat{P}^\omega), \delta_{(A,P)}^\omega \rangle$ , and on morphisms  $\phi : (A, P) \rightarrow (B, Q)$  in  $\mathbf{RBA}$  by  $(G_\omega)_{\mathbf{C}_\omega}(\phi) = \phi^\mathbb{N}$ . The natural transformations  $\varepsilon_{\mathbf{C}_\omega}^\omega$  and  $\eta_{\mathbf{C}_\omega}^\omega$  are defined similarly to  $\varepsilon^\omega$  and  $\eta^\omega$ , respectively.

Then there is a uniquely defined comultiplication functor  $H_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}_{\mathbf{C}_\omega}$  such that  $H_\omega G_\omega = (G_\omega)_{\mathbf{C}_\omega}$  and  $(V_\omega)_{\mathbf{C}_\omega} H_\omega = V_\omega$ .

Applying the dual of [26, Proposition 2.5(b)], we obtain

**Corollary 3.9.** *Let  $\omega \in \Omega$  satisfy Assumption 3.6. The comultiplication functor  $H_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}_{\mathbf{C}_\omega}$  is an isomorphism, i.e.,  $\mathbf{DRB}_\omega$  is comonadic over  $\mathbf{RBA}$ .*

**3.3. Lifting monads on DIF.** We let  $\mathbf{OA}_0$  denote the category of operated algebras  $(R, q)$  with the property  $q(\mathbf{1}_R) = 0$ , and let  $\mathbf{ORB}_\omega$  denote the category of type  $\omega$  operated Rota-Baxter algebras of weight  $\lambda$ . Thanks to Proposition 2.11, we obtain a functor

$$(29) \quad F_\omega : \mathbf{OA}_0 \rightarrow \mathbf{ORB}_\omega$$

given on objects  $(R, q)$  in  $\mathbf{OA}_0$  by  $F_\omega(R, q) = (\text{III}(R), \hat{q}^\omega, P_R)$  and on morphisms  $\varphi : (R, q) \rightarrow (S, d)$  in  $\mathbf{OA}_0$  by

$$F_\omega(\varphi) \left( \sum_{i=1}^k a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{i n_i} \right) = \sum_{i=1}^k \varphi(a_{i0}) \otimes \varphi(a_{i1}) \otimes \cdots \otimes \varphi(a_{i n_i}), \quad \sum_{i=1}^k a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{i n_i} \in \text{III}(R).$$

The category  $\mathbf{DRB}_\omega$ , which is a subcategory of  $\mathbf{ODA}_\omega$ , is also a subcategory of  $\mathbf{ORB}_\omega$ . We consider the following condition for a given  $\omega \in \Omega$  in Section 3.3:

**Assumption 3.10.** For every differential algebra  $(R, d)$  of weight  $\lambda$ , the extension  $\hat{d}^\omega$  of  $d$  to  $\text{III}(R)$  in Proposition 2.11 is a differential operator of weight  $\lambda$ .

This condition amounts to assuming that the functor  $F_\omega : \mathbf{OA}_0 \rightarrow \mathbf{ORB}_\omega$  in Eq. (29) restricts to a functor

$$F_\omega : \mathbf{DIF} \rightarrow \mathbf{DRB}_\omega.$$

Let  $U_\omega : \mathbf{ORB}_\omega \rightarrow \mathbf{OA}_0$  be the forgetful functor by forgetting the Rota-Baxter algebra structure. Then  $U_\omega$  restricts to a functor

$$U_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{DIF}.$$

**Lemma 3.11.** Let  $\omega \in \Omega$  satisfy Assumption 3.10 and  $(R, d, P) \in \mathbf{DRB}_\omega$ . Then we have

$$\varepsilon_{(R,P)} \hat{d}^\omega = d\varepsilon_{(R,P)},$$

where  $\varepsilon_{(R,P)}$  is given in Eq. (19).

*Proof.* We will prove by induction on  $n \in \mathbb{N}_+$  that

$$\varepsilon_{(R,P)} \hat{d}^\omega|_{\oplus_{i=1}^n R^{\otimes i}} = d\varepsilon_{(R,P)}|_{\oplus_{i=1}^n R^{\otimes i}}$$

holds.

First when  $n = 1$ ,

$$\varepsilon_{(R,P)} \hat{d}^\omega|_R = d = d\varepsilon_{(R,P)}|_R.$$

Next we assume that for a given  $k \in \mathbb{N}_+$ ,

$$\varepsilon_{(R,P)} \hat{d}^\omega|_{\oplus_{i=1}^k R^{\otimes i}} = d\varepsilon_{(R,P)}|_{\oplus_{i=1}^k R^{\otimes i}}$$

holds. Together with Eq. (18), we obtain

$$(30) \quad \varepsilon_{(R,P)} \phi(\hat{d}^\omega)|_{\oplus_{i=1}^k R^{\otimes i}} = \phi(d)\varepsilon_{(R,P)}|_{\oplus_{i=1}^k R^{\otimes i}}, \quad \varepsilon_{(R,P)} \psi(\hat{d}^\omega)|_{\oplus_{i=1}^k R^{\otimes i}} = \psi(d)\varepsilon_{(R,P)}|_{\oplus_{i=1}^k R^{\otimes i}}.$$

Let  $v = v_0 \otimes v' \in R^{\otimes(k+1)}$  with  $v' \in R^{\otimes k}$ . Since

$$\begin{aligned} & \varepsilon_{(R,P)}(\hat{d}^\omega(v)) \\ &= \varepsilon_{(R,P)}(d(v_0)P_R(v') + (v_0 + \lambda d(v_0))(\phi(\hat{d}^\omega) + P_R\psi(\hat{d}^\omega))(v')) \quad (\text{by Eq. (17)}) \\ &= d(v_0)P(\varepsilon_{(R,P)}(v')) + (v_0 + \lambda d(v_0))(\varepsilon_{(R,P)}\phi(\hat{d}^\omega))(v') \\ & \quad + (v_0 + \lambda d(v_0))(P(\varepsilon_{(R,P)}\psi(\hat{d}^\omega)))(v') \quad (\text{by Eq. (19)}) \\ &= d(v_0)P(\varepsilon_{(R,P)}(v')) + (v_0 + \lambda d(v_0))(\phi(d)\varepsilon_{(R,P)})(v') \\ & \quad + (v_0 + \lambda d(v_0))(P(\psi(d)\varepsilon_{(R,P)}))(v') \quad (\text{by Eq. (30)}) \end{aligned}$$

and

$$\begin{aligned} & d(\varepsilon_{(R,P)}(v)) \\ &= d(v_0 P(\varepsilon_{(R,P)}(v'))) \quad (\text{by Eq. (19)}) \\ &= d(v_0)P(\varepsilon_{(R,P)}(v')) + (v_0 + \lambda d(v_0))(dP)(\varepsilon_{(R,P)}(v')) \quad (\text{by Eq. (2)}) \\ &= d(v_0)P(\varepsilon_{(R,P)}(v')) + (v_0 + \lambda d(v_0))(\phi(d) + P\psi(d))(\varepsilon_{(R,P)}(v')) \quad (\text{by Eq. (10)}) \\ &= d(v_0)P(\varepsilon_{(R,P)}(v')) + (v_0 + \lambda d(v_0))(\phi(d)\varepsilon_{(R,P)})(v') + (v_0 + \lambda d(v_0))(P(\psi(d)\varepsilon_{(R,P)}))(v'), \end{aligned}$$

we obtain  $(\varepsilon_{(R,P)} \hat{d}^\omega)(v) = (d\varepsilon_{(R,P)})(v)$ . This completes the induction.  $\square$

**Proposition 3.12.** Let  $\omega \in \Omega$  satisfy Assumption 3.10. The functor  $F_\omega : \mathbf{DIF} \rightarrow \mathbf{DRB}_\omega$  is the left adjoint of the forgetful functor  $U_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{DIF}$ . In other words, there is an adjunction  $\langle F_\omega, U_\omega, \eta^\omega, \varepsilon^\omega \rangle : \mathbf{DIF} \rightarrow \mathbf{DRB}_\omega$ .

*Proof.* By [18], we just need to define natural transformations  $\eta^\omega : \mathbf{IDIF} \rightarrow U_\omega F_\omega$  and  $\varepsilon^\omega : F_\omega U_\omega \rightarrow \mathbf{IDRB}_\omega$  that satisfy the equations  $U_\omega \varepsilon^\omega \circ \eta^\omega U_\omega = U_\omega$  and  $\varepsilon^\omega F_\omega \circ F_\omega \eta^\omega = F_\omega$ .

For any  $(R, d) \in \mathbf{DIF}$ , we define

$$\eta_{(R,d)}^\omega : (R, d) \rightarrow (\mathbf{III}(R), \hat{d}^\omega)$$

to be the natural embedding map. Then  $\eta_{(R,d)}^\omega$  is an algebra homomorphism. Since

$$(\eta_{(R,d)}^\omega d)(r) = d(r) = (\hat{d}^\omega \eta_{(R,d)}^\omega)(r) \quad \text{for all } r \in R,$$

$\eta_{(R,d)}^\omega$  is a morphism in  $\mathbf{DIF}$ . As  $\eta_{(R,d)}^\omega$  is a natural embedding,  $\eta^\omega$  is a natural transformation.

For any  $(R, d, P) \in \mathbf{DRB}_\omega$ , define  $\varepsilon_{(R,d,P)}^\omega : (\mathbf{III}(R), \hat{d}^\omega, P_R) \rightarrow (R, d, P)$  by

$$\varepsilon_{(R,d,P)}^\omega \left( \sum_{i=1}^k a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \right) := \sum_{i=1}^k a_{i0} P(a_{i1} P(\cdots P(a_{in_i}) \cdots))$$

for any  $\sum_{i=1}^k a_{i0} \otimes a_{i1} \otimes \cdots \otimes a_{in_i} \in \mathbf{III}(R)$ . We can check that  $\varepsilon_{(R,d,P)}^\omega$  is a Rota-Baxter algebra homomorphism. Further,  $\varepsilon_{(R,d,P)}^\omega \hat{d}^\omega = d \varepsilon_{(R,d,P)}^\omega$  by Lemma 3.11 and  $\varepsilon_{(R,d,P)}^\omega = \varepsilon_{(R,P)}$ . Thus  $\varepsilon_{(R,d,P)}^\omega$  is a morphism in  $\mathbf{DRB}_\omega$ . If  $\varphi : (R, d, P) \rightarrow (R', d', P')$  is any morphism in  $\mathbf{DRB}_\omega$ , then  $\varepsilon_{(R',d',P')}^\omega (F_\omega U_\omega)(\varphi) = \varphi \varepsilon_{(R,d,P)}^\omega$ , that is,  $\varepsilon^\omega$  is a natural transformation.

To see that  $U_\omega \varepsilon^\omega \circ \eta^\omega U_\omega = U_\omega$ , let  $(R, D, P) \in \mathbf{DRB}_\omega$  and  $r \in R$ . Then

$$\varepsilon_{(R,D,P)}^\omega (\eta_{(R,D)}^\omega(r)) = \varepsilon_{(R,D,P)}^\omega(r) = r.$$

Similarly, to see that  $\varepsilon^\omega F_\omega \circ F_\omega \eta^\omega = F_\omega$ , let  $(A, d) \in \mathbf{DIF}$  and  $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in \mathbf{III}(A)$ . Then

$$\varepsilon_{(\mathbf{III}(A), \hat{d}^\omega, P_A)}^\omega ((F_\omega \eta_{(A,d)}^\omega)(a_0 \otimes a_1 \otimes \cdots \otimes a_n)) = \varepsilon_{(\mathbf{III}(A), \hat{d}^\omega, P_A)}^\omega (a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes a_1 \otimes \cdots \otimes a_n,$$

as needed.  $\square$

The adjunction  $\langle F_\omega, U_\omega, \eta^\omega, \varepsilon^\omega \rangle : \mathbf{DIF} \rightarrow \mathbf{DRB}_\omega$  gives rise to a monad  $\mathbf{T}_\omega = \langle T_\omega, \eta^\omega, \mu^\omega \rangle$  on  $\mathbf{DIF}$ , where  $T_\omega = U_\omega F_\omega : \mathbf{DIF} \rightarrow \mathbf{DIF}$  is a functor and  $\mu^\omega := U_\omega \varepsilon^\omega F_\omega : T_\omega T_\omega \rightarrow T_\omega$ . Note that the monad  $\mathbf{T}_\omega$  is a lifting of  $\mathbf{T}$  on  $\mathbf{DIF}$ .

As before, the monad  $\mathbf{T}_\omega$  induces a category of  $\mathbf{T}_\omega$ -algebras, denoted by  $\mathbf{DIF}^{\mathbf{T}_\omega}$ , and gives rise to an adjunction

$$\langle F_\omega^{\mathbf{T}_\omega}, U_\omega^{\mathbf{T}_\omega}, (\eta^\omega)^{\mathbf{T}_\omega}, (\varepsilon^\omega)^{\mathbf{T}_\omega} \rangle : \mathbf{DIF} \rightarrow \mathbf{DIF}^{\mathbf{T}_\omega},$$

where  $F_\omega^{\mathbf{T}_\omega} : \mathbf{DIF} \rightarrow \mathbf{DIF}^{\mathbf{T}_\omega}$  is given on objects  $(A, d)$  in  $\mathbf{DIF}$  by  $F_\omega^{\mathbf{T}_\omega}(A, d) = \langle (\mathbf{III}(A), \hat{d}^\omega), \mu_{(A,d)}^\omega \rangle$  and on morphisms  $\varphi : (A, d) \rightarrow (A', d')$  in  $\mathbf{DIF}$  by  $F_\omega^{\mathbf{T}_\omega}(\varphi) = F_\omega(\varphi)$ . The functor  $U_\omega^{\mathbf{T}_\omega} : \mathbf{DIF}^{\mathbf{T}_\omega} \rightarrow \mathbf{DIF}$  is defined on objects  $\langle (A, d), f \rangle$  in  $\mathbf{DIF}^{\mathbf{T}_\omega}$  by  $U_\omega^{\mathbf{T}_\omega} \langle (A, d), f \rangle = (A, d)$ , and on morphisms  $\varphi : \langle (A, d), f \rangle \rightarrow \langle (A', d'), f' \rangle$  in  $\mathbf{DIF}^{\mathbf{T}_\omega}$  by  $U_\omega^{\mathbf{T}_\omega}(\varphi) = \varphi$ . The natural transformations  $(\eta^\omega)^{\mathbf{T}_\omega}$  and  $(\varepsilon^\omega)^{\mathbf{T}_\omega}$  are defined similarly as  $\eta^\omega$  and  $\varepsilon^\omega$ , respectively. Then there is a uniquely defined comparison functor  $K_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{DIF}^{\mathbf{T}_\omega}$  such that  $K_\omega F_\omega = F_\omega^{\mathbf{T}_\omega}$  and  $U_\omega^{\mathbf{T}_\omega} K_\omega = U_\omega$ .

As a special case of [26, Proposition 2.5(b)], we obtain

**Corollary 3.13.** *Let  $\omega \in \Omega$  satisfy Assumption 3.10. The comparison functor  $K_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{DIF}^{\mathbf{T}_\omega}$  is an isomorphism, that is,  $\mathbf{DRB}_\omega$  is monadic over  $\mathbf{DIF}$ .*



**3.4. Mixed distributive laws.** We now give the equivalence among the existence of extensions in Section 2, of liftings of (co)monads in Sections 3.2 and 3.3, and mixed distributive laws involving type  $\omega$  differential Rota-Baxter algebras.

We first recall from [24] some background information on mixed distributive laws, a generalization of the notion of a distributive law introduced by J. Beck in his fundamental work [3].

**Definition 3.14.** Let a category  $\mathbf{A}$ , a monad  $\mathbf{T} = \langle T, \eta, \mu \rangle$  and a comonad  $\mathbf{C} = \langle C, \varepsilon, \delta \rangle$  on  $\mathbf{A}$  be given. A **mixed distributive law of  $\mathbf{T}$  over  $\mathbf{C}$**  is a natural transformation  $\beta : TC \rightarrow CT$  satisfying the following conditions.

- (i)  $\beta \circ \eta C = C\eta$ ;
- (ii)  $\varepsilon T \circ \beta = T\varepsilon$ ;
- (iii)  $\delta T \circ \beta = C\beta \circ \beta C \circ T\delta$ ;
- (iv)  $\beta \circ \mu C = C\mu \circ \beta T \circ T\beta$ .

**Theorem 3.15.** *The following statements are equivalent.*

- (i) *There exists a mixed distributive law of  $\mathbf{T}$  over  $\mathbf{C}$ .*
- (ii) *There exists a comonad  $\tilde{\mathbf{C}} = \langle \tilde{C}, \tilde{\varepsilon}, \tilde{\delta} \rangle$  on  $\mathbf{A}^{\mathbf{T}}$  which lifts  $\mathbf{C}$  (i.e.,  $U^{\mathbf{T}}\tilde{C} = CU^{\mathbf{T}}$ ,  $U^{\mathbf{T}}\tilde{\varepsilon} = \varepsilon U^{\mathbf{T}}$ ,  $U^{\mathbf{T}}\tilde{\delta} = \delta U^{\mathbf{T}}$ ).*
- (iii) *There exists a monad  $\tilde{\mathbf{T}} = \langle \tilde{T}, \tilde{\eta}, \tilde{\mu} \rangle$  on  $\mathbf{A}_{\mathbf{C}}$  which lifts  $\mathbf{T}$ .*

*Proof.* Theorem 2.2 and Remark 2.3 from [23] give the correspondence among lifting  $\mathcal{V}$ -comonads, lifting  $\mathcal{V}$ -monads and  $\mathcal{V}$ -mixed distributive laws, where  $\mathcal{V}$  is a symmetric monoidal closed category. The theorem follows as a special case when  $\mathcal{V}$  is taken to be the category of sets.  $\square$

Recall  $\Omega = \{xy - (\phi(x) + y\psi(x)) \mid \phi, \psi \in \mathbf{k}[x]\}$  as in Eq. (9). Now we have arrived at the first main result of this paper.

**Theorem 3.16.** *Let  $\omega \in \Omega$  be given. The following statements are equivalent:*

- (i) *For every Rota-Baxter operator  $P$  on every algebra  $R$ , the unique coextension  $\widehat{P}^{\omega}$  of  $P$  to  $R^{\mathbb{N}}$  given in Proposition 2.7 is a Rota-Baxter operator.*
- (ii) *For every differential operator  $d$  on every algebra  $R$ , the unique extension  $\hat{d}^{\omega}$  of  $d$  to  $\text{III}(R)$  given in Proposition 2.11 is a differential operator.*
- (iii) *The functor  $G_{\omega} : \mathbf{OA} \rightarrow \mathbf{ODA}_{\omega}$  in Eq. (23) restricts to a functor  $G_{\omega} : \mathbf{RBA} \rightarrow \mathbf{DRB}_{\omega}$ .*
- (iv) *The functor  $F_{\omega} : \mathbf{OA}_0 \rightarrow \mathbf{ORB}_{\omega}$  in Eq. (29) restricts to a functor  $F_{\omega} : \mathbf{DIF} \rightarrow \mathbf{DRB}_{\omega}$ .*
- (v) *There is a lifting comonad  $\tilde{\mathbf{C}} = \langle \tilde{C}, \tilde{\varepsilon}, \tilde{\delta} \rangle$  of  $\mathbf{C}$  on  $\mathbf{RBA}$  and an isomorphism  $\tilde{H} : \mathbf{DRB}_{\omega} \rightarrow \mathbf{RBA}_{\tilde{\mathbf{C}}}$  over  $\mathbf{RBA}$  given by  $\tilde{H}(R, d, P) = \langle (R, P), \theta_{(R, d, P)} : (R, P) \rightarrow (R^{\mathbb{N}}, \tilde{P}) \rangle$ , where  $\tilde{C}(R, P) := (R^{\mathbb{N}}, \tilde{P})$  and  $\theta_{(R, d, P)}(u)_n := d^n(u)$  for all  $u \in R, n \in \mathbb{N}$ .*
- (vi) *There is a lifting monad  $\tilde{\mathbf{T}} = \langle \tilde{T}, \tilde{\eta}, \tilde{\mu} \rangle$  of  $\mathbf{T}$  on  $\mathbf{DIF}$  and an isomorphism  $\tilde{K} : \mathbf{DRB}_{\omega} \rightarrow \mathbf{DIF}^{\tilde{\mathbf{T}}}$  over  $\mathbf{DIF}$  given by  $\tilde{H}(R, d, P) = \langle (R, d), \vartheta_{(R, d, P)} : (\text{III}(R), \tilde{d}) \rightarrow (R, d) \rangle$ , where  $\tilde{T}(R, d) := (\text{III}(R), \tilde{d})$  and  $\vartheta_{(R, d, P)}(v_0 \otimes v_1 \otimes \cdots \otimes v_m) := v_0 P(v_1 P(\cdots P(v_m) \cdots))$  for all  $v_0 \otimes v_1 \otimes \cdots \otimes v_m \in \text{III}(R)$ .*
- (vii) *There is a mixed distributive law  $\beta : TC \rightarrow CT$  such that  $(\mathbf{ALG}_{\mathbf{C}})^{\tilde{\mathbf{T}}_{\beta}}$  is isomorphic to the category  $\mathbf{DRB}_{\omega}$ , where  $\tilde{\mathbf{T}}_{\beta}$  is a lifting monad of  $\mathbf{T}$  given by the mixed distributive law  $\beta$ .*

*Proof.* (i)  $\implies$  (iii). Item (i) combined with Proposition 2.7 gives  $(R^{\mathbb{N}}, \partial_R, \widehat{P}^{\omega}) \in \mathbf{DRB}_{\omega}$ .

(ii)  $\implies$  (iv). Similarly, Item (ii) combined with Proposition 2.11 gives  $(\text{III}(R), \hat{d}^{\omega}, P_R) \in \mathbf{DRB}_{\omega}$ .

(iii)  $\implies$  (v). In Proposition 3.8, we obtain an adjunction  $\langle V_\omega, G_\omega, \eta^\omega, \varepsilon^\omega \rangle : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}$  which gives a lifting comonad  $\mathbf{C}_\omega$  of  $\mathbf{C}$  on  $\mathbf{RBA}$ . Furthermore, there is an isomorphism  $H_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{RBA}_{\mathbf{C}_\omega}$  in Corollary 3.9, as required.

(iv)  $\implies$  (vi). In Proposition 3.12, we obtain an adjunction  $\langle F_\omega, U_\omega, \eta^\omega, \varepsilon^\omega \rangle : \mathbf{DIF} \rightarrow \mathbf{DRB}_\omega$  which gives a lifting monad  $\mathbf{T}_\omega$  of  $\mathbf{T}$  on  $\mathbf{DIF}$ . Furthermore, there is an isomorphism  $K_\omega : \mathbf{DRB}_\omega \rightarrow \mathbf{DIF}^{\mathbf{T}_\omega}$  in Corollary 3.13, as required.

(v)  $\implies$  (i). The lifting comonad  $\tilde{\mathbf{C}} = \langle \tilde{C}, \tilde{\varepsilon}, \tilde{\delta} \rangle$  on  $\mathbf{RBA}$  induces an adjunction

$$\langle V_{\tilde{\mathbf{C}}}, G_{\tilde{\mathbf{C}}}, \eta_{\tilde{\mathbf{C}}}, \varepsilon_{\tilde{\mathbf{C}}} \rangle : \mathbf{RBA}_{\tilde{\mathbf{C}}} \rightarrow \mathbf{RBA}.$$

Since  $\tilde{\varepsilon}_{(R,P)} : (R^\mathbb{N}, \tilde{P}) \rightarrow (R, P)$  is a morphism in  $\mathbf{RBA}$ ,  $\tilde{\varepsilon}_{(R,P)}(\tilde{P}(f)) = P(\tilde{\varepsilon}_{(R,P)}(f))$  for all  $f \in R^\mathbb{N}$ . That is,  $\tilde{P}_0(f) = P(f_0)$ . Then  $\tilde{P}$  is a coextension of  $P$ . Recall that  $G_{\tilde{\mathbf{C}}}(R, P) = \langle (R^\mathbb{N}, \tilde{P}), \tilde{\delta}_{(R,P)} \rangle \in \mathbf{RBA}_{\tilde{\mathbf{C}}}$  and  $\tilde{H}^{-1} \langle (R^\mathbb{N}, \tilde{P}), \tilde{\delta}_{(R,P)} \rangle = (R^\mathbb{N}, \partial_R, \tilde{P}) \in \mathbf{DRB}_\omega$ . Then  $\partial_R \tilde{P} = \phi(\partial_R) + \tilde{P}\psi(\partial_R)$ . By the uniqueness of the coextension in Proposition 2.7, we have  $\tilde{P} = \tilde{P}^\omega$ . Therefore,  $\tilde{P}^\omega$  is a Rota-Baxter operator.

(vi)  $\implies$  (ii). The lifting monad  $\tilde{\mathbf{T}} = \langle \tilde{T}, \tilde{\eta}, \tilde{\mu} \rangle$  on  $\mathbf{DIF}$  induces an adjunction

$$\langle F^{\tilde{\mathbf{T}}}, U^{\tilde{\mathbf{T}}}, \eta^{\tilde{\mathbf{T}}}, \varepsilon^{\tilde{\mathbf{T}}} \rangle : \mathbf{DIF} \rightarrow \mathbf{DIF}^{\tilde{\mathbf{T}}}.$$

Since  $\tilde{\eta}_{(R,d)} : (R, d) \rightarrow (\text{III}(R), \tilde{d})$  is a morphism in  $\mathbf{DIF}$ , we have  $\tilde{d}\tilde{\eta}_{(R,d)} = \tilde{\eta}_{(R,d)}d$ . That is,  $\tilde{d}(r) = d(r)$  for all  $r \in R$ . Then  $\tilde{d}$  is an extension of  $d$ . Recall that  $F^{\tilde{\mathbf{T}}}(R, d) = \langle (\text{III}(R), \tilde{d}), \tilde{\mu}_{(R,d)} \rangle \in \mathbf{DIF}^{\tilde{\mathbf{T}}}$  and  $\tilde{K}^{-1} \langle (\text{III}(R), \tilde{d}), \tilde{\mu}_{(R,d)} \rangle = (\text{III}(R), \tilde{d}, P_R) \in \mathbf{DRB}_\omega$ . Let  $u = u_0 \otimes u' \in R^{\otimes(n+1)}$  with  $u' \in R^{\otimes n}$ . Since  $\tilde{d}$  is a differential operator of weight  $\lambda$  on  $\text{III}(R)$ , we have

$$\tilde{d}(u) = \tilde{d}(u_0 P_R(u')) = \tilde{d}(u_0) P_R(u') + (u_0 + \lambda \tilde{d}(u_0))(\tilde{d} P_R(u')) \quad (\text{by Eq. (2)}).$$

Also by  $\tilde{d} P_R = \phi(\tilde{d}) + P_R \psi(\tilde{d})$ , we have  $\tilde{d}(u) = d(u_0) \otimes u' + (u_0 + \lambda d(u_0))(\phi(\tilde{d}) + P_R \psi(\tilde{d}))(u')$ . Further, we obtain  $\tilde{d}(\oplus_{i=1}^n R^{\otimes i}) \subseteq \oplus_{i=1}^n R^{\otimes i}$  which is proved by induction on  $n \in \mathbb{N}_+$ . That is,  $\tilde{d}$  satisfies Eqs. (17) and (18). Then by the uniqueness of the extension in Proposition 2.11, we have  $\tilde{d} = \hat{d}^\omega$ . Therefore,  $\hat{d}^\omega$  is a differential operator.

(v)  $\iff$  (vi)  $\iff$  (vii). This follows directly from Theorem 3.15, [23, Theorem 2.4] and [26, Theorem 5.3].  $\square$

#### 4. CONDITIONS OF THE COEXTENSIONS

In view of Theorem 3.16, it is important to classify in concrete terms the elements  $\omega \in \Omega$  that satisfy the equivalent conditions in the theorem. We obtain two results in this direction, characterizing the elements  $\omega \in \Omega$  satisfying condition (i) in Theorem 3.16 first in the case when the weight is zero and then in the case when the weight is generic (namely for all weights). As we can see, the condition imposed on  $\omega$  is very strict.

In this section, assume that  $\mathbf{k}$  is a domain of characteristic 0 as noted in the Introduction.

As in Eq. (9), let  $\Omega := xy + \mathbf{k}[x] + y\mathbf{k}[x]$ . Consider the following subsets of  $\Omega$ :

$$\Omega_0 := \{xy - a_0 \mid a_0 \in \mathbf{k}\} \cup \{xy - (b_0 y + yx) \mid b_0 \in \mathbf{k}\}, \quad \Omega_{\mathbf{k}} := \{xy, xy - 1, xy - yx\}.$$

**Theorem 4.1.** *Let  $\omega \in \Omega$  be given.*

(i) *The following statements are equivalent.*

- (a) *For every Rota-Baxter algebra  $(R, P)$  of weight 0, the coextension  $\widehat{P}^\omega$  of  $P$  to the differential algebra  $(R^\mathbb{N}, \partial_R)$  of weight 0 given in Proposition 2.7 is again a Rota-Baxter operator of weight 0;*

- (b)  $\omega$  is in  $\Omega_0$ .
- (ii) The following statements are equivalent.
  - (a) For every Rota-Baxter algebra  $(R, P)$  of every weight  $\lambda$ , the coextension  $\widehat{P}^\omega$  of  $P$  to the differential algebra  $(R^\mathbb{N}, \partial_R)$  of weight  $\lambda$  given in Proposition 2.7 is again a Rota-Baxter operator of weight  $\lambda$ ;
  - (b)  $\omega$  is in  $\Omega_k$ .

*Proof.* (Summary) The proof is divided into four parts. Let  $\omega = xy - (\phi(x) + y\psi(x))$  with  $\phi, \psi \in \mathbf{k}[x]$ . The first three parts cover the proof of Item (i), partitioned into the following three cases of  $\omega$ .

**Case 1.**  $\psi = 0$ . This is proved in Section 4.1;

**Case 2.**  $\psi \neq 0$  and  $\phi = 0$ . This is proved in Section 4.2;

**Case 3.**  $\psi \neq 0$  and  $\phi \neq 0$ . This is proved in Section 4.3.

The fourth part, given in Section 4.4, proves Item (ii) of the theorem.  $\square$

The proof of Theorem 4.1.(i) will apply the following special case of Proposition 2.8.(ii) repeatedly. Since  $\lambda = 0$ , Eq. (16) becomes

$$\sum_{j=0}^n \binom{n}{j} \widehat{Q}_{n-j}(f) \widehat{Q}_j(g) = \widehat{Q}_n(\widehat{Q}(f)g) + \widehat{Q}_n(f\widehat{Q}(g)) \quad \text{for all } f, g \in R^\mathbb{N}, n \in \mathbb{N}.$$

As a consequence, we have

**Corollary 4.2.** Let  $(R, P)$  be a Rota-Baxter algebra of weight 0. If there are  $f, g \in R^\mathbb{N}$  such that

$$(31) \quad (\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) \neq 0,$$

then the coextension  $\widehat{P}^\omega$  of  $P$  to  $(R^\mathbb{N}, \partial_R)$  is not a Rota-Baxter operator of weight 0.

For a given  $\omega = xy - (\phi(x) + y\psi(x)) \in \Omega$  with  $\phi(x) := \sum_{i=0}^r a_i x^i$ ,  $\psi(x) := \sum_{j=0}^s b_j x^j$ , and each  $f \in R^\mathbb{N}$ ,  $n \in \mathbb{N}_+$ , we obtain

$$\begin{aligned} \widehat{P}_n^\omega(f) &= (\phi(\partial_R)_{n-1} + \widehat{P}_{n-1}^\omega \psi(\partial_R))(f) \quad (\text{by Eq. (13)}) \\ &= \left( \left( \sum_{i=0}^r a_i \partial_R^i \right)_{n-1} + \sum_{j=0}^s b_j \widehat{P}_{n-1}^\omega \partial_R^j \right)(f) \\ (32) \quad &= \sum_{i=0}^r a_i f_{n-1+i} + \sum_{j=0}^s b_j \widehat{P}_{n-1}^\omega(\partial_R^j f). \end{aligned}$$

Recall from Example 2.2 that, for  $m \in \mathbb{N}_+$ ,  $\overline{P_k}$  is a Rota-Baxter operator on the quotient algebra  $\mathbb{III}(\mathbf{k})/I_m$ . These Rota-Baxter algebras  $(\mathbb{III}(\mathbf{k})/I_m, \overline{P_k})$  will be used with Corollary 4.2 to give counterexamples in the later proofs.

**4.1. Proof of Theorem 4.1.(i): Case 1.** In this case  $\omega := xy - \phi(x) \in \Omega$ , where  $\phi \in \mathbf{k}[x]$ . Thus to prove Case 1 of Theorem 4.1.(i), we only need to prove the following proposition which provides an additional equivalent condition.

**Proposition 4.3.** Let  $\omega := xy - \phi(x)$  with  $\phi(x) := \sum_{i=0}^r a_i x^i$ . The following statements are equivalent.

- (i) For every Rota-Baxter algebra  $(R, P)$  of weight 0, the coextension  $\widehat{P}^\omega$  of  $P$  to the differential algebra  $(R^\mathbb{N}, \partial_R)$  of weight 0 is again a Rota-Baxter operator of weight 0;
- (ii)  $\phi = a_0$ , that is,  $\omega = xy - a_0$ ;
- (iii) For every Rota-Baxter algebra  $(R, P)$  of weight 0, we have

$$(33) \quad \widehat{P}^\omega(f) = (P(f_0), a_0 f_0, a_0 f_1, \dots) \quad \text{for all } f \in R^\mathbb{N}.$$

*Proof.* By Eq. (32), we have

$$(34) \quad \widehat{P}_n^\omega(f) = \sum_{i=0}^r a_i f_{n-1+i} \quad \text{for all } f \in R^\mathbb{N}, n \in \mathbb{N}_+.$$

In particular, when  $n = 1$ ,

$$(35) \quad \widehat{P}_1^\omega(f) = \sum_{i=0}^r a_i f_i \quad \text{for all } f \in R^\mathbb{N}.$$

(ii)  $\implies$  (iii). When  $\omega = xy - a_0$ , applying Eq. (34), we obtain  $\widehat{P}_n^\omega(f) = a_0 f_{n-1}$  for all  $f \in R^\mathbb{N}, n \in \mathbb{N}_+$ . Together with  $\widehat{P}_0^\omega(f) = P(f_0)$  from the definition of a coextension, Eq. (33) follows.

(iii)  $\implies$  (ii). Suppose  $r := \deg \phi \geq 1$ , so  $a_r \neq 0$ . Take the Rota-Baxter algebra  $(R, P) := (\text{III}(\mathbf{k})/I_1, \overline{P}_k)$  in Example 2.2, and  $f := (f_\ell) \in (\text{III}(\mathbf{k})/I_1)^\mathbb{N}$  with  $f_\ell := \delta_{\ell, r} \overline{z_0}$ . Then Eq. (35) gives  $\widehat{P}_1^\omega(f) = \sum_{i=0}^r a_i \delta_{i, r} \overline{z_0} = a_r \overline{z_0} \neq \overline{0}$  while Eq. (33) gives  $\widehat{P}_1^\omega(f) = a_0 f_0 = a_0 \delta_{0, r} \overline{z_0} = \overline{0}$ . This is a contradiction. Therefore,  $\phi = a_0$ .

(i)  $\implies$  (ii). We just need to show that if  $r := \deg \phi \geq 1$ , then there is a Rota-Baxter algebra  $(R, P)$  such that the coextension  $\widehat{P}^\omega$  of  $P$  is not a Rota-Baxter operator on  $R^\mathbb{N}$ . When  $r \geq 1$ , we see  $a_r \neq 0$ . Let  $M_n$  denote the maximum of the subscripts  $m$  of the expressions  $f_m$  appearing on the right hand side of Eq. (34). Then  $M_n = n - 1 + r$ . Take  $(R, P) := (\text{III}(\mathbf{k})/I_2, \overline{P}_k)$  in Example 2.2, and  $f := (f_\ell) \in (\text{III}(\mathbf{k})/I_2)^\mathbb{N}$  with  $f_\ell := \delta_{\ell, M_r} \overline{z_0} = \delta_{\ell, 2r-1} \overline{z_0}$ . For each  $n \in \mathbb{N}_+$  with  $n \leq r$ , Eq. (34) becomes

$$\widehat{P}_n^\omega(f) = \sum_{i=0}^r a_i \delta_{n-1+i, 2r-1} \overline{z_0} = a_r \delta_{n-1+r, 2r-1} \overline{z_0} = \begin{cases} a_r \overline{z_0}, & \text{if } n = r, \\ \overline{0}, & \text{if } 1 \leq n < r. \end{cases}$$

Also by  $\widehat{P}_0^\omega(f) = P(f_0) = \overline{0}$ , we have

$$(36) \quad \widehat{P}_r^\omega(f) = a_r \overline{z_0}, \quad \widehat{P}_n^\omega(f) = \overline{0} \quad \text{for each } n \in \mathbb{N} \text{ with } n < r.$$

Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_2)^\mathbb{N}$  with  $g_k := \delta_{k, 0} \overline{z_0}$ , i.e.,  $g$  is the identity element of  $(\text{III}(\mathbf{k})/I_2)^\mathbb{N}$ . Then by Eqs. (35) and (36), we obtain

$$(37) \quad \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = \widehat{P}_1^\omega(\widehat{P}^\omega(f)) = \sum_{i=0}^r a_i \widehat{P}_i^\omega(f) = a_r \widehat{P}_r^\omega(f) = a_r^2 \overline{z_0}.$$

Since  $r \leq 2r - 1$ , we have  $f_i = \delta_{i, 2r-1} \overline{z_0} = \overline{0}$  for each  $i < r$ . Then Eq. (35) gives  $\widehat{P}_1^\omega(f) = \sum_{i=0}^r a_i f_i = a_r f_r$ . Then we obtain

$$(38) \quad \widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g) = a_r f_r \widehat{P}_0^\omega(g).$$

By Eq. (35),  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \sum_{i=0}^r a_i(f\widehat{P}^\omega(g))_i$ . So applying Eq. (7), we have

$$\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \sum_{i=0}^r a_i \sum_{j=0}^i \binom{i}{j} f_j \widehat{P}_{i-j}^\omega(g).$$

Applying  $f_j = \bar{0}$  for each  $j < r$  again, we obtain

$$(39) \quad \widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = a_r f_r \widehat{P}_0^\omega(g).$$

Combining Eqs. (37), (38) and (39), we obtain

$$\begin{aligned} (\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) &= (a_r^2 \bar{z}_0 + a_r f_r \widehat{P}_0^\omega(g)) - a_r f_r \widehat{P}_0^\omega(g) \\ &= a_r^2 \bar{z}_0 \neq \bar{0}. \end{aligned}$$

Thus by Corollary 4.2,  $\widehat{P}^\omega$  is not a Rota-Baxter operator on  $R^\mathbb{N}$ .

(ii)  $\implies$  (i). By Proposition 2.8.(ii), we need to show that for any Rota-Baxter algebra  $(R, P)$ , and all  $f, g \in R^\mathbb{N}$ ,  $n \in \mathbb{N}$ ,

$$\sum_{k=0}^n \binom{n}{k} \widehat{P}_k^\omega(f) \widehat{P}_{n-k}^\omega(g) = \widehat{P}_n^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_n^\omega(f\widehat{P}^\omega(g))$$

holds. Applying Proposition 2.8.(i), we have

$$\widehat{P}_0^\omega(f)\widehat{P}_0^\omega(g) = \widehat{P}_0^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_0^\omega(f\widehat{P}^\omega(g)).$$

Since  $\omega = xy - a_0$ , Eq. (34) gives

$$(40) \quad \widehat{P}_n^\omega(h) = a_0 h_{n-1} \quad \text{for all } h \in R^\mathbb{N}, n \in \mathbb{N}_+.$$

Then

$$\begin{aligned} \widehat{P}_n^\omega(\widehat{P}^\omega(f)g) &= a_0(\widehat{P}^\omega(f)g)_{n-1} \quad (\text{by Eq. (40)}) \\ &= \widehat{P}_0^\omega(f)(a_0 g_{n-1}) + \sum_{k=1}^{n-1} \binom{n-1}{k} \widehat{P}_k^\omega(f)(a_0 g_{n-k-1}) \quad (\text{by Eq. (7)}) \\ (41) \quad &= \widehat{P}_0^\omega(f)\widehat{P}_n^\omega(g) + \sum_{k=1}^{n-1} \binom{n-1}{k} \widehat{P}_k^\omega(f)\widehat{P}_{n-k}^\omega(g). \quad (\text{by Eq. (40)}) \end{aligned}$$

Exchanging  $f$  and  $g$ , and then applying the commutativity of the multiplication, we obtain

$$(42) \quad \widehat{P}_n^\omega(f\widehat{P}^\omega(g)) = \widehat{P}_n^\omega(f)\widehat{P}_0^\omega(g) + \sum_{k=1}^{n-1} \binom{n-1}{k} \widehat{P}_{n-k}^\omega(f)\widehat{P}_k^\omega(g) = \widehat{P}_n^\omega(f)\widehat{P}_0^\omega(g) + \sum_{k=1}^{n-1} \binom{n-1}{n-k} \widehat{P}_k^\omega(f)\widehat{P}_{n-k}^\omega(g)$$

by exchanging  $k$  and  $n-k$ . Combining Eqs. (41) and (42), and  $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{n-k}$ , we obtain

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \widehat{P}_k^\omega(f) \widehat{P}_{n-k}^\omega(g) &= \widehat{P}_0^\omega(f)\widehat{P}_n^\omega(g) + \widehat{P}_n^\omega(f)\widehat{P}_0^\omega(g) + \sum_{k=1}^{n-1} \binom{n}{k} \widehat{P}_k^\omega(f) \widehat{P}_{n-k}^\omega(g) \\ &= \widehat{P}_n^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_n^\omega(f\widehat{P}^\omega(g)), \end{aligned}$$

as needed.  $\square$

**4.2. Proof of Theorem 4.1.(i): Case 2.** In this case  $\omega := xy - y\psi(x) \in \Omega$  with  $\psi \in \mathbf{k}[x]$ . Thus to prove the Case 2 of Theorem 4.1.(i), we only need to prove the following strengthened form.

**Proposition 4.4.** *Let  $\omega := xy - y\psi(x)$  with  $\psi(x) := \sum_{j=0}^s b_j x^j \neq 0$ . The following statements are equivalent.*

- (i) *For every Rota-Baxter algebra  $(R, P)$  of weight 0, the coextension  $\widehat{P}^\omega$  of  $P$  to the differential algebra  $(R^\mathbb{N}, \partial_R)$  of weight 0 is again a Rota-Baxter operator of weight 0;*
- (ii)  *$\deg \psi = 1$  and  $b_1 = 1$ , that is,  $\omega = xy - (b_0 y + yx)$ ;*
- (iii) *For every Rota-Baxter algebra  $(R, P)$  of weight 0 and each  $f \in R^\mathbb{N}$ , we have*

$$\widehat{P}^\omega(f) = (\widehat{P}_0^\omega(f), \widehat{P}_1^\omega(f), \dots, \widehat{P}_n^\omega(f), \dots),$$

where  $\widehat{P}_0^\omega(f) = P(f_0)$  and for each  $n \in \mathbb{N}_+$ ,  $\widehat{P}_n^\omega(f)$  is given recursively by

$$(43) \quad \widehat{P}_n^\omega(f) = b_0 \widehat{P}_{n-1}^\omega(f) + \widehat{P}_{n-1}^\omega(\partial_R f).$$

In particular, if  $b_0 = 0$ , then  $\widehat{P}_n^\omega(f) = \widehat{P}_{n-1}^\omega(\partial_R f) = \dots = \widehat{P}_0^\omega(\partial_R^n f) = P(f_n)$ . That is,

$$\widehat{P}^\omega(f) = (P(f_0), P(f_1), P(f_2), \dots).$$

*Proof.* Recall from Eq. (32) that the coextension  $\widehat{P}^\omega$  is given by

$$(44) \quad \widehat{P}_n^\omega(f) = \sum_{j=0}^s b_j \widehat{P}_{n-1}^\omega(\partial_R^j f) \quad \text{for all } f \in R^\mathbb{N}, n \in \mathbb{N}_+.$$

In particular,

$$(45) \quad \widehat{P}_1^\omega(f) = \sum_{j=0}^s b_j \widehat{P}_0^\omega(\partial_R^j f) = \sum_{j=0}^s b_j P(f_j) \quad \text{for all } f \in R^\mathbb{N}.$$

In general, by iterating Eq. (44), we obtain

$$(46) \quad \widehat{P}_n^\omega(f) = \sum_{j_1=0}^s b_{j_1} \sum_{j_2=0}^s b_{j_2} \dots \sum_{j_n=0}^s b_{j_n} \widehat{P}_0^\omega(\partial_R^{j_1+j_2+\dots+j_n} f) = \sum_{j_1, j_2, \dots, j_n=0}^s b_{j_1} b_{j_2} \dots b_{j_n} P(f_{j_1+j_2+\dots+j_n}).$$

(ii)  $\implies$  (iii). For any  $f \in R^\mathbb{N}$ ,  $\widehat{P}_0^\omega(f) = P(f_0)$  follows from the definition of a coextension. By  $\omega := xy - (b_0 y + yx)$  and Eq. (44), we obtain  $\widehat{P}_n^\omega(f) = b_0 \widehat{P}_{n-1}^\omega(f) + \widehat{P}_{n-1}^\omega(\partial_R f)$  for all  $n \in \mathbb{N}_+$ .

(iii)  $\implies$  (ii). Assume that Item (iii) holds. Suppose  $s := \deg \psi \geq 2$ . Take  $(R, P) := (\text{III}(\mathbf{k})/I_2, \overline{P}_{\mathbf{k}})$  in Example 2.2. Let  $f := (f_k) \in (\text{III}(\mathbf{k})/I_2)^\mathbb{N}$  with  $f_k := \delta_{k,s} \overline{z_0}$ . Then Eq. (43) gives

$$\widehat{P}_1^\omega(f) = b_0 \widehat{P}_0^\omega(f) + \widehat{P}_0^\omega(\partial_R f) = b_0 P(f_0) + P(f_1) = b_0 P(\delta_{0,s} \overline{z_0}) + P(\delta_{1,s} \overline{z_0}) = \overline{0}$$

while  $\widehat{P}_1^\omega(f) = \sum_{j=0}^s b_j P(\delta_{j,s} \overline{z_0}) = b_s \overline{z_1} \neq \overline{0}$  by Eq. (45). This is a contradiction. Thus  $s = \deg \psi \leq 1$ .

Now take  $f := (f_\ell) \in (\text{III}(\mathbf{k})/I_2)^\mathbb{N}$  with  $f_\ell := \delta_{\ell,1} \overline{z_0}$ . Then Eq. (43) gives

$$\widehat{P}_1^\omega(f) = b_0 \widehat{P}_0^\omega(f) + \widehat{P}_0^\omega(\partial_R f) = b_0 P(f_0) + P(f_1) = b_0 P(\delta_{0,1} \overline{z_0}) + P(\delta_{1,1} \overline{z_0}) = \overline{z_1}$$

while Eq. (45) gives

$$\widehat{P}_1^\omega(f) = \sum_{j=0}^s b_j P(\delta_{j,1} \overline{z_0}) = \begin{cases} b_1 P(\overline{z_0}) = b_1 \overline{z_1}, & \text{if } s = 1, \\ \overline{0}, & \text{if } s = 0. \end{cases}$$

Thus we obtain  $s = 1$  and  $\overline{z_1} = b_1 \overline{z_1}$ . Then  $b_1 = 1$  since  $\overline{z_1}$  is one of the basis elements. Therefore,  $\omega = xy - (b_0 y + yx)$ .

(i)  $\implies$  (ii). Let  $s := \deg \psi$ . Consider  $(R, P) := (\text{III}(\mathbf{k})/I_3, \overline{P_k})$  and take  $g := (g_k) \in (\text{III}(\mathbf{k})/I_3)^\mathbb{N}$  with  $g_k := \delta_{k,0} \overline{z_0}$ , i.e.,  $g$  is the identity element of  $(\text{III}(\mathbf{k})/I_3)^\mathbb{N}$ . Then  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = \widehat{P}_1^\omega(\widehat{P}^\omega(f))$ . So applying Eq. (45), we obtain

$$(47) \quad \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = \sum_{j=0}^s b_j P(\widehat{P}_j^\omega(f)).$$

Suppose  $s = 0$ , i.e.,  $\psi = b_0$ . Then Eq. (47) becomes  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = b_0 P(\widehat{P}_0^\omega(f))$ . Let  $f := (f_\ell) \in (\text{III}(\mathbf{k})/I_3)^\mathbb{N}$  with  $f_\ell := \delta_{\ell,0} \overline{z_0}$ . Applying  $f = g$  and the commutativity of the multiplication, we have

$$(48) \quad \widehat{P}_1^\omega(f \widehat{P}^\omega(g)) = \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = b_0 P(P(f_0)) = b_0 \overline{z_2}.$$

Applying Eqs. (5) and (45), we obtain

$$(49) \quad \widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g) = P(f_0) b_0 P(g_0) + b_0 P(f_0) P(g_0) = 2b_0 \overline{z_1}^2 = 4b_0 \overline{z_2}.$$

Combining Eqs. (48) and (49) gives

$$(\widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g)) - (\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f \widehat{P}^\omega(g))) = 4b_0 \overline{z_2} - 2b_0 \overline{z_2} = 2b_0 \overline{z_2} \neq \overline{0}.$$

So  $\widehat{P}^\omega$  is not a Rota-Baxter operator on  $R^\mathbb{N}$  by Corollary 4.2. So we must have  $s \geq 1$ .

Now let  $s \geq 1$  be given. Then  $b_s \neq 0$ . Let  $M_n$  denote the maximum of the subscripts  $m$  of the expressions  $f_m$  appearing on the right hand side of Eq. (46):

$$M_n := \max\{j_1 + j_2 + \cdots + j_n \mid 0 \leq j_1, j_2, \dots, j_n \leq s\} = ns.$$

Take  $f := (f_\ell) \in (\text{III}(\mathbf{k})/I_3)^\mathbb{N}$  with  $f_\ell := \delta_{\ell, M_s} \overline{z_0} = \delta_{\ell, s^2} \overline{z_0}$ . For each  $n \in \mathbb{N}_+$  with  $n \leq s$ , Eq. (46) becomes

$$\widehat{P}_n^\omega(f) = \sum_{j_1, j_2, \dots, j_n=0}^s b_{j_1} b_{j_2} \cdots b_{j_n} P(\delta_{j_1+j_2+\dots+j_n, s^2} \overline{z_0}) = b_s^n P(\delta_{ns, s^2} \overline{z_0}) = \begin{cases} b_s^n P(\overline{z_0}) = b_s^n \overline{z_1}, & \text{if } n = s, \\ \overline{0}, & \text{if } 1 \leq n < s. \end{cases}$$

Together with  $\widehat{P}_0^\omega(f) = P(f_0) = P(\delta_{0, s^2} \overline{z_0}) = \overline{0}$  from  $s^2 > 0$ , we obtain

$$(50) \quad \widehat{P}_s^\omega(f) = b_s^s \overline{z_1}, \quad \widehat{P}_n^\omega(f) = \overline{0} \quad \text{for each } n \in \mathbb{N} \text{ with } n < s.$$

Then Eq. (47) gives

$$(51) \quad \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = b_s P(\widehat{P}_s^\omega(f)) = b_s P(b_s^s \overline{z_1}) = b_s^{s+1} \overline{z_2}.$$

Also by Eqs. (45) and (7), we have

$$(52) \quad \widehat{P}_1^\omega(f \widehat{P}^\omega(g)) = \sum_{j=0}^s b_j P((f \widehat{P}^\omega(g))_j) = \sum_{j=0}^s b_j P\left(\sum_{i=0}^j \binom{j}{i} f_i \widehat{P}_{j-i}^\omega(g)\right) = \begin{cases} \overline{0}, & \text{if } s \geq 2, \\ b_1 \overline{z_2}, & \text{if } s = 1. \end{cases}$$



Here the last equation follows from  $f_i = \delta_{i,s^2} \overline{z_0}$  since  $0 \leq i \leq j \leq s \leq s^2$  with equality holding in the last inequality if and only if  $s = 1$ . Further applying Eq. (50), we have

$$(53) \quad \widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g) = \begin{cases} \overline{0}, & \text{if } s \geq 2, \\ b_1 \overline{z_1} \widehat{P}_0^\omega(g) = b_1 \overline{z_1}^2 = 2b_1 \overline{z_2}, & \text{if } s = 1. \end{cases}$$

Combining Eqs. (51), (52) and (53), we obtain

$$\left( \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f \widehat{P}^\omega(g)) \right) - \left( \widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g) \right) = \begin{cases} b_s^{s+1} \overline{z_2} \neq \overline{0}, & \text{if } s \geq 2, \\ b_1(b_1 - 1) \overline{z_2} & \text{if } s = 1. \end{cases}$$

Then by Corollary 4.2, when  $s \geq 2$ ,  $\widehat{P}^\omega$  is not a Rota-Baxter operator. When  $s = 1$ , we obtain  $b_1 - 1 = 0$ , i.e.,  $b_1 = 1$ .

Therefore, we must have  $s = 1$  and  $b_1 = 1$ .

(ii)  $\implies$  (i). Let  $(R, P)$  be an arbitrary Rota-Baxter algebra  $(R, P)$ . We will prove that  $\widehat{P}^\omega$  is a Rota-Baxter operator on  $R^\mathbb{N}$  by verifying the componentwise formulation

$$(54) \quad (\widehat{P}^\omega(f) \widehat{P}^\omega(g))_n = \widehat{P}_n^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_n^\omega(f \widehat{P}^\omega(g)) \quad \text{for all } f, g \in R^\mathbb{N}, n \in \mathbb{N},$$

of the Rota-Baxter relation in Eq. (3). We will carry out the verification by induction on  $n$ .

First by Proposition 2.8.(i), we have

$$(\widehat{P}^\omega(f) \widehat{P}^\omega(g))_0 = \widehat{P}_0^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_0^\omega(f \widehat{P}^\omega(g)).$$

Assume that for a given  $k \in \mathbb{N}$ , Eq. (54) holds. Then we derive

$$\begin{aligned} & \widehat{P}_{k+1}^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_{k+1}^\omega(f \widehat{P}^\omega(g)) \\ &= (b_0 \widehat{P}_k^\omega + \widehat{P}_k^\omega \partial_R)(\widehat{P}^\omega(f)g + f \widehat{P}^\omega(g)) \quad (\text{by Eq. (44)}) \\ &= b_0 \widehat{P}_k^\omega(\widehat{P}^\omega(f)g + f \widehat{P}^\omega(g)) + \widehat{P}_k^\omega((\partial_R \widehat{P}^\omega)(f)g + \widehat{P}^\omega(f) \partial_R(g)) \\ & \quad + \widehat{P}_k^\omega(\partial_R(f) \widehat{P}^\omega(g) + f(\partial_R \widehat{P}^\omega)(g)) \quad (\text{by Eq. (1)}) \\ &= b_0 \widehat{P}_k^\omega(\widehat{P}^\omega(f)g + f \widehat{P}^\omega(g)) + \widehat{P}_k^\omega((b_0 \widehat{P}^\omega + \widehat{P}^\omega \partial_R)(f)g + \widehat{P}^\omega(f) \partial_R(g)) \\ & \quad + \widehat{P}_k^\omega(\partial_R(f) \widehat{P}^\omega(g) + f(b_0 \widehat{P}^\omega + \widehat{P}^\omega \partial_R)(g)) \quad (\text{by Eq. (12)}) \\ &= 2b_0 (\widehat{P}^\omega(f) \widehat{P}^\omega(g))_k + (\widehat{P}^\omega(\partial_R f) \widehat{P}^\omega(g))_k + (\widehat{P}^\omega(f) \widehat{P}^\omega(\partial_R g))_k \quad (\text{by the induction hypothesis}) \\ &= ((b_0 \widehat{P}^\omega + \widehat{P}^\omega \partial_R)(f) \widehat{P}^\omega(g))_k + (\widehat{P}^\omega(f)(b_0 \widehat{P}^\omega + \widehat{P}^\omega \partial_R)(g))_k \\ &= ((\partial_R \widehat{P}^\omega)(f) \widehat{P}^\omega(g) + \widehat{P}^\omega(f)(\partial_R \widehat{P}^\omega)(g))_k \quad (\text{by Eq. (12)}) \\ &= (\widehat{P}^\omega(f) \widehat{P}^\omega(g))_{k+1} \quad (\text{by Eq. (8)}). \end{aligned}$$

This completes the induction.  $\square$

**4.3. Proof of Theorem 4.1.(i): Case 3.** In this case,  $\omega := xy - (\phi(x) + y\psi(x)) \in \Omega$ , where  $\phi, \psi \in \mathbf{k}[x]$  are nonzero with  $r := \deg \phi, s := \deg \psi \in \mathbb{N}$ . To prove Theorem 4.1.(i) in this case, we will apply the same idea as in the previous two cases, namely by taking the maximum of the subscripts. But in order for the idea to work, we need to partition  $\mathbb{N}^2$  into eight subsets before carrying out the proof in Proposition 4.6.

Let  $(R, P)$  denote an arbitrary Rota-Baxter algebra, and  $\phi(x) := \sum_{i=0}^r a_i x^i$  and  $\psi(x) := \sum_{j=0}^s b_j x^j$ . As in Eq. (32), we have

$$(55) \quad \widehat{P}_n^\omega(f) = \sum_{i=0}^r a_i f_{n-1+i} + \sum_{j=0}^s b_j \widehat{P}_{n-1}^\omega(\partial_R^j f) \quad \text{for all } f \in R^\mathbb{N}, n \in \mathbb{N}_+.$$

In particular, if  $n = 1$ , then Eq. (55) becomes

$$(56) \quad \widehat{P}_1^\omega(f) = \sum_{i=0}^r a_i f_i + \sum_{j=0}^s b_j P(f_j).$$

Expanding the recursion in Eq. (55), we obtain

$$\begin{aligned} \widehat{P}_n^\omega(f) &= \sum_{i=0}^r a_i f_{n-1+i} + \sum_{j_1=0}^s b_{j_1} \widehat{P}_{n-1}^\omega(\partial_R^{j_1} f) \quad (\text{by Eq. (55)}) \\ &= \sum_{i=0}^r a_i f_{n-1+i} + \sum_{j_1=0}^s b_{j_1} \left( \sum_{i=0}^r a_i f_{n-2+i+j_1} + \sum_{j_2=0}^s b_{j_2} \widehat{P}_{n-2}^\omega(\partial_R^{j_1+j_2} f) \right) \quad (\text{by Eq. (55)}) \\ &= \sum_{i=0}^r a_i f_{n-1+i} + \sum_{i=0}^r \sum_{j_1=0}^s a_i b_{j_1} f_{n-2+i+j_1} + \sum_{j_1, j_2=0}^s b_{j_1} b_{j_2} \widehat{P}_{n-2}^\omega(\partial_R^{j_1+j_2} f). \end{aligned}$$

Repeating this process leads to

$$(57) \quad \widehat{P}_n^\omega(f) = \sum_{i=0}^r \sum_{k=0}^{n-1} \sum_{j_1, \dots, j_k=0}^s a_i b_{j_1} \cdots b_{j_k} f_{n-1-k+i+j_1+\dots+j_k} + \sum_{j_1, \dots, j_n=0}^s b_{j_1} \cdots b_{j_n} P(f_{j_1+\dots+j_n})$$

for all  $f \in R^\mathbb{N}$ ,  $n \in \mathbb{N}_+$ .

Let  $M_n$  denote the maximum of the subscripts of the expressions  $f_m$  appearing on the right hand side of Eq. (57):

$$M_n := \max \{n-1-k+i+j_1+\dots+j_k, j_1+\dots+j_n \mid 0 \leq k \leq n-1, 0 \leq i \leq r, 0 \leq j_1, \dots, j_n \leq s\}.$$

By first partitioning  $s \in \mathbb{N}$  into  $s > 1$ ,  $s = 1$  and  $s < 1$  (that is  $s = 0$ ) and then partitioning each of the three cases into the subcases of  $r > s$ ,  $r = s$  and  $r < s$  (the latter subcase is valid only when  $s > 1$  and  $s = 1$ ), we partition  $(r, s) \in \mathbb{N}^2$  into eight cases in the following lemma.

**Lemma 4.5.** *Let  $n \in \mathbb{N}_+$  and  $f := (f_\ell) \in R^\mathbb{N}$  with  $f_\ell := \delta_{\ell, M_n} u$ , where  $u$  is a given nonzero element in  $R$ . The possibilities of  $M_n$  and  $\widehat{P}_\sigma^\omega(f)$  for all  $\sigma \leq n$  are as follows.*

- (i) If  $s > 1$  and  $r > s$ , then  $M_n = r + (n-1)s$ ,  $\widehat{P}_n^\omega(f) = a_r b_s^{n-1} u$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;
- (ii) If  $s > 1$  and  $r = s$ , then  $M_n = ns$ ,  $\widehat{P}_n^\omega(f) = a_r b_s^{n-1} u + b_s^n P(u)$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;
- (iii) If  $s > 1$  and  $r < s$ , then  $M_n = ns$ ,  $\widehat{P}_n^\omega(f) = b_s^n P(u)$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;
- (iv) If  $s = 1$  and  $r > s$ , then  $M_n = n-1+r$ ,  $\widehat{P}_n^\omega(f) = \sum_{k=0}^{n-1} a_r b_1^k u$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;
- (v) If  $s = 1$  and  $r = s$ , then  $M_n = n$ ,  $\widehat{P}_n^\omega(f) = \sum_{k=0}^{n-1} a_r b_1^k u + b_1^n P(u)$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;
- (vi) If  $s = 1$  and  $r < s$ , then  $M_n = n$ ,  $\widehat{P}_n^\omega(f) = b_1^n P(u)$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;
- (vii) If  $s = 0$  and  $r > s$ , then  $M_n = n-1+r$ ,  $\widehat{P}_n^\omega(f) = a_r u$  and  $\widehat{P}_\sigma^\omega(f) = 0$  for  $\sigma < n$ ;

(viii) If  $s = 0$  and  $r = s$ , then  $M_n = n - 1$ ,  $\widehat{P}_n^\omega(f) = a_r u + \delta_{n,1} b_0 P(u)$ ,  $\widehat{P}_\sigma^\omega(f) = \delta_{n,1} P(u)$  for  $\sigma < n$ .

*Proof.* By the choice of  $f$ , Eq. (57) becomes

$$\widehat{P}_n^\omega(f) = \sum_{i=0}^r \sum_{k=0}^{n-1} \sum_{j_1, \dots, j_k=0}^s a_i b_{j_1} \cdots b_{j_k} \delta_{n-1-k+i+j_1+\dots+j_k, M_n} u + \sum_{j_1, \dots, j_n=0}^s b_{j_1} \cdots b_{j_n} P(\delta_{j_1+\dots+j_n, M_n} u).$$

Since the two indices of the Kronecker deltas are possibly equal only when  $i$  and  $j_1, \dots, j_n$  are maximized, we have

$$\begin{aligned} \widehat{P}_n^\omega(f) &= \sum_{k=0}^{n-1} a_r b_s^k \delta_{n-1-k+r+ks, M_n} u + b_s^n P(\delta_{ns, M_n} u) \\ (58) \quad &= \sum_{k=0}^{n-1} a_r b_s^k \delta_{n-1+r+k(s-1), M_n} u + b_s^n \delta_{n-1+s+(n-1)(s-1), M_n} P(u). \end{aligned}$$

We first prove the first and second equations in all the cases of the lemma.

When  $s > 1$ , namely  $s - 1 > 0$ , then by maximizing  $k$ , Eq. (58) becomes

$$\widehat{P}_n^\omega(f) = a_r b_s^{n-1} \delta_{n-1+r+(n-1)(s-1), M_n} u + b_s^n \delta_{n-1+s+(n-1)(s-1), M_n} P(u) = a_r b_s^{n-1} \delta_{r+(n-1)s, M_n} u + b_s^n \delta_{s+(n-1)s, M_n} P(u).$$

Thus when  $r > s$  (resp.  $r = s$ , resp.  $r < s$ ), we obtain  $M_n = r + (n - 1)s$  (resp.  $M_n = ns$ , resp.  $M_n = ns$ ) and  $\widehat{P}_n^\omega(f) = a_r b_s^{n-1} u$  (resp.  $\widehat{P}_n^\omega(f) = a_r b_s^{n-1} u + b_s^n P(u)$ , resp.  $\widehat{P}_n^\omega(f) = b_s^n P(u)$ ), proving the first and second equations in cases (i) – (iii) of the lemma.

When  $s = 1$ , namely  $s - 1 = 0$ , then Eq. (58) becomes

$$\widehat{P}_n^\omega(f) = \sum_{k=0}^{n-1} a_r b_1^k \delta_{n-1+r, M_n} u + b_1^n \delta_{n-1+s, M_n} P(u).$$

Thus when  $r > s = 1$  (resp.  $r = s = 1$ , resp.  $r < s = 1$ ), we obtain  $M_n = n - 1 + r$  (resp.  $M_n = n$ , resp.  $M_n = n$ ) and  $\widehat{P}_n^\omega(f) = \sum_{k=0}^{n-1} a_r b_1^k u$  (resp.  $\widehat{P}_n^\omega(f) = \sum_{k=0}^{n-1} a_r b_1^k u + b_1^n P(u)$ , resp.  $\widehat{P}_n^\omega(f) = b_1^n P(u)$ ), proving the first and second equations in Item (iv) – (vi) of the lemma.

When  $s < 1$ , namely  $s = 0$  and  $s - 1 = -1$ , then minimizing  $k$ , Eq. (58) becomes

$$\widehat{P}_n^\omega(f) = a_r \delta_{n-1+r, M_n} u + b_0^n \delta_{0, M_n} P(u).$$

Thus when  $r > s = 0$  (resp.  $r = s = 0$ ), we obtain  $M_n = n - 1 + r$  (resp.  $M_n = n - 1$ ) and  $\widehat{P}_n^\omega(f) = a_r u$  (resp.  $\widehat{P}_n^\omega(f) = a_r u + \delta_{n,1} b_0 P(u)$ ), proving the first and second equations in Item (vii) – (viii) of the lemma.

Now we prove the third equations in all the cases of the lemma. In each of the cases (i) – (vii), since  $M_n > 0$ , we have  $\widehat{P}_0^\omega(f) = P(f_0) = P(\delta_{0, M_n}) = 0$ . For case (viii),  $\widehat{P}_0^\omega(f) = P(f_0) = P(\delta_{0, M_n} u) = P(\delta_{0, n-1} u) = \delta_{n,1} P(u)$ . This proves the third equations when  $\sigma = 0$ .

In each of the cases (i) – (viii), take  $\sigma$  with  $1 \leq \sigma < n$ . Then  $n > 1$  and so  $M_\sigma < M_n$ . Thus the expressions  $f_\tau$  appearing in  $\widehat{P}_\sigma^\omega(f)$  all vanish since the subscripts of the expressions are strictly smaller than  $M_n$ . Therefore,  $\widehat{P}_\sigma^\omega(f) = 0$ . This completes the proof of Lemma 4.5.  $\square$

We also need the following facts to proceed.

For a Rota-Baxter algebra  $(R, P)$ , take  $f := (f_\ell)$  and  $g := (g_k)$  in  $R^{\mathbb{N}}$ . Then

$$\begin{aligned}
 \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) &= \sum_{i=0}^r a_i(\widehat{P}^\omega(f)g)_i + \sum_{j=0}^s b_j P((\widehat{P}^\omega(f)g)_j) \quad (\text{by Eq. (56)}) \\
 (59) \quad &= \sum_{i=0}^r a_i \sum_{\sigma=0}^i \binom{i}{\sigma} \widehat{P}_\sigma^\omega(f) g_{i-\sigma} + \sum_{j=0}^s b_j P\left(\sum_{\tau=0}^j \binom{j}{\tau} \widehat{P}_\tau^\omega(f) g_{j-\tau}\right) \quad (\text{by Eq. (7)})
 \end{aligned}$$

and

$$\begin{aligned}
 \widehat{P}_1^\omega(f\widehat{P}^\omega(g)) &= \sum_{i=0}^r a_i(f\widehat{P}^\omega(g))_i + \sum_{j=0}^s b_j P((f\widehat{P}^\omega(g))_j) \quad (\text{by Eq. (56)}) \\
 (60) \quad &= \sum_{i=0}^r a_i \sum_{\sigma=0}^i \binom{i}{\sigma} f_\sigma \widehat{P}_{i-\sigma}^\omega(g) + \sum_{j=0}^s b_j P\left(\sum_{\tau=0}^j \binom{j}{\tau} f_\tau \widehat{P}_{j-\tau}^\omega(g)\right). \quad (\text{by Eq. (7)})
 \end{aligned}$$

Let  $N_f$  denote the maximal subscript of expressions  $f_m$  appearing in the right hand side of Eq. (60):

$$(61) \quad N_f := \max\{\sigma, \tau \mid \sigma \leq i \leq r, \tau \leq j \leq s\} = \max\{r, s\}.$$

Now take  $(R, P) := (\text{III}(\mathbf{k})/I_m, \overline{P}_{\mathbf{k}})$  from Example 2.2. Let  $g = (g_k)$  with  $g_k := \delta_{k,0}\overline{z}_0$ , i.e.,  $g$  is the identity element of  $(\text{III}(\mathbf{k})/I_m)^{\mathbb{N}}$ . Then by Eq. (56),

$$(62) \quad \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = \widehat{P}_1^\omega(\widehat{P}^\omega(f)) = \sum_{i=0}^r a_i \widehat{P}_i^\omega(f) + \sum_{j=0}^s b_j P(\widehat{P}_j^\omega(f)).$$

Eq. (56) also gives

$$\widehat{P}_1^\omega(g) = a_0 g_0 + b_0 P(g_0) = a_0 \overline{z}_0 + b_0 \overline{z}_1.$$

Then

$$(63) \quad \widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \widehat{P}_0^\omega(f)(a_0 \overline{z}_0 + b_0 \overline{z}_1) + \widehat{P}_1^\omega(f)\overline{z}_1.$$

Now we are ready to prove Case 3 of Theorem 4.1.(i).

**Proposition 4.6.** *For each  $\omega := xy - (\phi(x) + y\psi(x)) \in \Omega$  with nonzero  $\phi, \psi \in \mathbf{k}[x]$ , there is a Rota-Baxter algebra  $(R, P)$  of weight 0 such that the coextension  $\widehat{P}^\omega$  of  $P$  to  $(R^{\mathbb{N}}, \partial_R)$  is not a Rota-Baxter operator.*

*Proof.* By Corollary 4.2, we only need to prove that, for each given  $\omega$  as in the proposition, there is a Rota-Baxter algebra  $(R, P)$  and  $f, g \in R^{\mathbb{N}}$  such that

$$(64) \quad (\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) \neq 0.$$

We will divide the proof into the eight cases of  $r := \deg \phi$  and  $s := \deg \psi$  as in Lemma 4.5.

Denote  $\phi(x) := \sum_{i=0}^r a_i x^i$  and  $\psi(x) := \sum_{j=0}^s b_j x^j$ . So  $a_r, b_s \neq 0$ .

**Case (i).**  $s > 1, r > s$ . In Lemma 4.5.(i), take  $(R, P) := (\text{III}(\mathbf{k})/I_1, \overline{P}_{\mathbf{k}})$ ,  $n := r$  and  $u := \overline{z}_0$ . Then the lemma gives  $M_r = r + (r-1)s$ ,  $\widehat{P}_r^\omega(f) = a_r b_s^{r-1} \overline{z}_0$  and  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < r$ . Let  $g \in (\text{III}(\mathbf{k})/I_1)^{\mathbb{N}}$  be the identity element. Since  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < r$  and  $r > 2$  by the assumption of Case (i), Eqs. (62) and (63) give  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_r \widehat{P}_r^\omega(f)$  and  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ , respectively.

Also in this case,  $N_f = \max\{r, s\} = r$  in Eq. (61). So we have  $N_f < M_r$ . Then by  $f_\ell = \delta_{\ell, M_r} \overline{z_0}$ , Eq. (60) gives  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \overline{0}$ . Thus we obtain

$$\left(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))\right) - \left(\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)\right) = a_r \widehat{P}_r^\omega(f) = a_r^2 b_s^{r-1} \overline{z_0} \neq \overline{0}.$$

This is what we need.

We use the similar argument as in Case (i) to prove other cases as follows.

**Case (ii).**  $s > 1, r = s$ . In Lemma 4.5.(ii), taking  $(R, P) := (\text{III}(\mathbf{k})/I_2, \overline{P_k})$ ,  $n := s$  and  $u := \overline{z_1}$  gives  $M_s = s^2$ ,  $\widehat{P}_s^\omega(f) = a_r b_s^{s-1} \overline{z_1} + b_s^s P(\overline{z_1}) = a_r b_s^{s-1} \overline{z_1}$  and  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < s$ . Let  $g \in (\text{III}(\mathbf{k})/I_2)^\mathbb{N}$  be the identity. Then by Eqs. (62) and (63), we have  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_r \widehat{P}_r^\omega(f) + b_s P(\widehat{P}_s^\omega(f)) = a_r^2 b_s^{s-1} \overline{z_1}$  and  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ . Further by  $N_f = s < M_s$  and  $f_\ell = \delta_{\ell, M_s} \overline{z_1}$ , Eq. (60) becomes  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \overline{0}$ . Thus we obtain

$$\left(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))\right) - \left(\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)\right) = a_r^2 b_s^{s-1} \overline{z_1} \neq \overline{0}.$$

**Case (iii).**  $s > 1, r < s$ . In Lemma 4.5.(iii), take  $(R, P) := (\text{III}(\mathbf{k})/I_3, \overline{P_k})$ ,  $n := s$  and  $u := \overline{z_0}$ . Then  $M_s = s^2$ ,  $\widehat{P}_s^\omega(f) = b_s^s P(\overline{z_0}) = b_s^s \overline{z_1}$  and  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < s$ . Let  $g \in (\text{III}(\mathbf{k})/I_3)^\mathbb{N}$  be the identity. By Eqs. (62) and (63), we have  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = b_s P(\widehat{P}_s^\omega(f))$  and  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ . Since  $N_f = s < M_s$  and  $f_\ell = \delta_{\ell, M_s} \overline{z_0}$ , Eq. (60) becomes  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \overline{0}$ . Thus we obtain

$$\left(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))\right) - \left(\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)\right) = b_s P(\widehat{P}_s^\omega(f)) = b_s^{s+1} \overline{z_2} \neq \overline{0}.$$

**Case (iv).**  $s = 1, r > s$ . We consider  $(R, P) := (\text{III}(\mathbf{k})/I_1, \overline{P_k})$  and divide the proof into two subcases depending on whether or not  $\sum_{k=0}^{r-1} b_1^k$  is zero.

First assume  $\sum_{k=0}^{r-1} b_1^k \neq 0$ . In Lemma 4.5.(iv), take  $n := r$  and  $u := \overline{z_0}$ . Then  $M_r = 2r - 1$ ,

$\widehat{P}_r^\omega(f) = \sum_{k=0}^{r-1} a_r b_1^k \overline{z_0}$  and  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < r$ . Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_1)^\mathbb{N}$  be the identity. Then Eqs. (62) and (63) give  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_r \widehat{P}_r^\omega(f)$  and  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ , respectively. Further, by  $N_f = r < M_r$  and  $f_\ell = \delta_{\ell, M_r} \overline{z_0}$ , Eq. (60) gives  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \overline{0}$ . Thus we obtain

$$\left(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))\right) - \left(\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)\right) = a_r \widehat{P}_r^\omega(f) = a_r^2 \left(\sum_{k=0}^{r-1} b_1^k\right) \overline{z_0} \neq 0.$$

Next assume  $\sum_{k=0}^{r-1} b_1^k = 0$ . Then  $\sum_{k=0}^{r-2} b_1^k = -b_1^{r-1}$ . In Lemma 4.5.(iv), take  $n := r - 1$  and  $u := \overline{z_0}$ . Then  $M_{r-1} = 2(r - 1)$ ,  $\widehat{P}_{r-1}^\omega(f) = \sum_{k=0}^{r-2} a_r b_1^k \overline{z_0} = a_r \left(\sum_{k=0}^{r-2} b_1^k\right) \overline{z_0} = -a_r b_1^{r-1} \overline{z_0}$  and  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < r - 1$ . Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_1)^\mathbb{N}$  with  $g_k := \delta_{k,1} \overline{z_0}$ . Then by Eq. (59), we have  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_r \binom{r}{r-1} \widehat{P}_{r-1}^\omega(f) g_1 = -r a_r^2 b_1^{r-1} \overline{z_0}$ . Further, by  $N_f = r \leq M_{r-1}$  and  $f_\ell = \delta_{\ell, M_{r-1}} \overline{z_0}$ , Eq. (60) gives  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = a_r f_r \widehat{P}_0^\omega(g) = a_r f_r P(g_0) = \overline{0}$ . By  $\widehat{P}_0^\omega(f) = \widehat{P}_0^\omega(g) = \overline{0}$ , we have  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ . Thus we obtain

$$\left(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))\right) - \left(\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)\right) = -r a_r^2 b_1^{r-1} \overline{z_0} \neq \overline{0}.$$

**Case (v).**  $s = 1, r = s$ . In Lemma 4.5.(v), take  $(R, P) := (\text{III}(\mathbf{k})/I_1, \overline{P_k})$ ,  $n := 1$  and  $u := \overline{z_0}$ . Then  $M_1 = 1$ ,  $\widehat{P}_1^\omega(f) = a_1 \overline{z_0} + b_1 P(\overline{z_0}) = a_1 \overline{z_0}$  and  $\widehat{P}_0^\omega(f) = \overline{0}$ . Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_1)^\mathbb{N}$

with  $g_k := \delta_{k,0}\overline{z_0}$ . So  $g$  is the identity. By Eq. (62) and  $\widehat{P}_0^\omega(f) = \overline{0}$ , we have  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_1\widehat{P}_1^\omega(f) + b_1P(\widehat{P}_1^\omega(f))$ . Since  $f_\ell = \delta_{\ell,1}\overline{z_0}$  and  $\widehat{P}_0^\omega(g) = P(g_0) = \overline{0}$ , Eq. (60) gives  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \overline{0}$ . By Eq. (63) and  $\widehat{P}_0^\omega(f) = \overline{0}$ , we have  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ . Thus we obtain

$$(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) = a_1\widehat{P}_1^\omega(f) + b_1P(\widehat{P}_1^\omega(f)) = a_1^2\overline{z_0} \neq \overline{0}.$$

**Case (vi).**  $s = 1, r < s$ . We consider  $(R, P) := (\text{III}(\mathbf{k})/I_3, \overline{P_k})$  and divide the proof into two subcases depending on whether or not  $b_1 = 1$ .

First assume  $b_1 \neq 1$ . In Lemma 4.5.(vi), take  $n := 1$  and  $u := \overline{z_0}$ . Then  $M_1 = 1$ ,  $\widehat{P}_1^\omega(f) = b_1P(\overline{z_0}) = b_1\overline{z_1}$  and  $\widehat{P}_0^\omega(f) = \overline{0}$ . Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_3)^\mathbb{N}$  be the identity, so  $g_k := \delta_{k,0}\overline{z_0}$ . Then Eq. (62) becomes  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = b_1P(\widehat{P}_1^\omega(f))$ . By  $f_\ell = \delta_{\ell,1}\overline{z_0}$ , Eq. (60) gives  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = b_1P(f_1\widehat{P}_0^\omega(g))$ . By  $\widehat{P}_0^\omega(f) = \overline{0}$ , Eq. (63) becomes  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \widehat{P}_1^\omega(f)\overline{z_1}$ . Thus we obtain

$$(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) = b_1^2\overline{z_2} + b_1\overline{z_2} - b_1\overline{z_1}^2 = b_1(b_1 - 1)\overline{z_2} \neq \overline{0}.$$

Next assume  $b_1 = 1$ . Let both  $f := (f_\ell)$  and  $g := (g_k)$  be the identity element of  $(\text{III}(\mathbf{k})/I_3)^\mathbb{N}$ . Then  $\widehat{P}_0^\omega(f) = \widehat{P}_0^\omega(g) = \overline{z_1}$ . By Eq. (56), we have  $\widehat{P}_1^\omega(f) = \widehat{P}_1^\omega(g) = a_0\overline{z_0} + b_0\overline{z_1}$ . Then applying the commutativity of the multiplication and Eq. (62), we have

$$\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_0\widehat{P}_0^\omega(f) + b_0P(\widehat{P}_0^\omega(f)) + P(\widehat{P}_1^\omega(f)) = 2a_0\overline{z_1} + 2b_0\overline{z_2}.$$

Further,  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = 2\overline{z_1}(a_0\overline{z_0} + b_0\overline{z_1}) = 2a_0\overline{z_1} + 4b_0\overline{z_2}$ . Thus we obtain

$$(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) = 2a_0\overline{z_1} \neq \overline{0}.$$

**Case (vii).**  $s = 0, r > s$ . In Lemma 4.5.(vii), take  $(R, P) := (\text{III}(\mathbf{k})/I_1, \overline{P_k})$ ,  $n := r$  and  $u := \overline{z_0}$ . Then  $M_r = 2r - 1$ ,  $\widehat{P}_r^\omega(f) = a_r\overline{z_0}$  and  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < r$ . Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_1)^\mathbb{N}$  be the identity with  $g_k := \delta_{k,0}\overline{z_0}$ . Since  $\widehat{P}_\sigma^\omega(f) = \overline{0}$  for  $\sigma < r$ , Eq. (62) gives  $\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_r\widehat{P}_r^\omega(f)$ . By  $N_f = r \leq M_r$  and  $f_\ell = \delta_{\ell,M_r}\overline{z_0}$ , Eq. (60) gives  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = a_rf_r\widehat{P}_0^\omega(g) = a_rf_rP(g_0) = \overline{0}$ . By  $\widehat{P}_0^\omega(f) = \overline{0}$ , Eq. (63) becomes  $\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g) = \overline{0}$ . Thus we obtain

$$(\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) = a_r\widehat{P}_r^\omega(f) = a_r^2\overline{z_0} \neq \overline{0}.$$

**Case (viii).**  $s = 0, r = s$ . In Lemma 4.5.(viii), take  $(R, P) := (\text{III}(\mathbf{k})/I_3, \overline{P_k})$ ,  $n := 1$  and  $u := \overline{z_0}$ . Then  $M_1 = 0$ ,  $\widehat{P}_1^\omega(f) = a_0\overline{z_0} + b_0P(\overline{z_0}) = a_0\overline{z_0} + b_0\overline{z_1}$  and  $\widehat{P}_0^\omega(f) = P(\overline{z_0}) = \overline{z_1}$ . Let  $g := (g_k) \in (\text{III}(\mathbf{k})/I_3)^\mathbb{N}$  with  $g_k := \delta_{k,0}\overline{z_0}$ , i.e.,  $g=f$ . Then applying the commutativity of the multiplication and Eq. (62), we have  $\widehat{P}_1^\omega(f\widehat{P}^\omega(g)) = \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) = a_0\widehat{P}_0^\omega(f) + b_0P(\widehat{P}_0^\omega(f))$ . Thus we obtain

$$\begin{aligned} (\widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g))) - (\widehat{P}_0^\omega(f)\widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f)\widehat{P}_0^\omega(g)) &= 2(a_0\overline{z_1} + b_0\overline{z_2}) - 2\overline{z_1}(a_0\overline{z_0} + b_0\overline{z_1}) \\ &= -2b_0\overline{z_2} \neq \overline{0}. \end{aligned}$$

To recapitulate, applying Corollary 4.2, we obtain that for each  $(s, r) \in \mathbb{N} \times \mathbb{N}$ , the given coextension  $\widehat{P}^\omega$  of the chosen  $P$  to  $(R^\mathbb{N}, \partial_R)$  is not a Rota-Baxter operator, completing the proof of Proposition 4.6.  $\square$

4.4. **Proof of Theorem 4.1.(ii).** Finally we prove Theorem 4.1.(ii).

Let  $(R, P)$  be an arbitrary Rota-Baxter algebra of arbitrary weight  $\lambda$ . Recall from Proposition 2.8.(ii) that the coextension  $\widehat{P}^\omega$  of  $P$  to  $(R^\mathbb{N}, \partial_R)$  is again a Rota-Baxter operator of weight  $\lambda$  if and only if for all  $f, g \in R^\mathbb{N}$ , and  $n \in \mathbb{N}$ ,

$$(65) \quad \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \widehat{P}_{n-j}^\omega(f) \widehat{P}_{k+j}^\omega(g) - \left( \widehat{P}_n^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_n^\omega(f\widehat{P}^\omega(g)) + \lambda \widehat{P}_n^\omega(fg) \right) = 0.$$

(iia)  $\implies$  (iib). If Item (iia) holds, then as a special case, for every Rota-Baxter algebra  $(R, P)$  of weight 0, the coextension  $\widehat{P}^\omega$  of  $P$  is still a Rota-Baxter operator of weight 0. So by Theorem 4.1.(i),  $\omega$  is in  $\Omega_0$ , that is,  $\omega = xy - a_0$  or  $\omega = xy - (b_0y + yx)$ .

First consider  $\omega = xy - a_0$ . Then Eq. (32) gives  $\widehat{P}_n^\omega(f) = a_0 f_{n-1}$  for all  $f \in R^\mathbb{N}$ ,  $n \in \mathbb{N}_+$ . Together with  $\widehat{P}_0^\omega(f) = P(f_0)$ , we obtain

$$(66) \quad \widehat{P}^\omega(f) = (P(f_0), a_0 f_0, a_0 f_1, \dots).$$

We take  $(R, P) := (\text{III}(\mathbf{k})/I_1, \overline{P_k})$  of weight  $\lambda$ , and  $f := (f_\ell), g := (g_k) \in (\text{III}(\mathbf{k})/I_1)^\mathbb{N}$  with  $f_\ell := \delta_{\ell,0} \overline{z_0}$  and  $g = f$ . Applying Eq. (66), we have

$$(67) \quad \widehat{P}_1^\omega(f) = \widehat{P}_1^\omega(g) = a_0 \overline{z_0}, \quad \widehat{P}_0^\omega(f) = \widehat{P}_0^\omega(g) = P(\overline{z_0}) = \overline{0}.$$

Then we obtain

$$\begin{aligned} \sum_{k=0}^1 \sum_{j=0}^{1-k} \binom{1}{k} \binom{1-k}{j} \lambda^k \widehat{P}_{1-j}^\omega(f) \widehat{P}_{k+j}^\omega(g) &= \widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g) + \lambda \widehat{P}_1^\omega(f) \widehat{P}_1^\omega(g) \\ &= \lambda a_0^2 \overline{z_0} \quad (\text{by Eq. (67)}) \end{aligned}$$

and

$$\begin{aligned} \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g)) + \lambda \widehat{P}_1^\omega(fg) &= a_0 \widehat{P}_0^\omega(f)g_0 + a_0 f_0 \widehat{P}_0^\omega(g) + \lambda a_0 f_0 g_0 \quad (\text{by Eq. (66)}) \\ &= \lambda a_0 \overline{z_0}. \quad (\text{by Eq. (67)}) \end{aligned}$$

Then

$$\sum_{k=0}^1 \sum_{j=0}^{1-k} \binom{1}{k} \binom{1-k}{j} \lambda^k \widehat{P}_{1-j}^\omega(f) \widehat{P}_{k+j}^\omega(g) - \left( \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g)) + \lambda \widehat{P}_1^\omega(fg) \right) = \lambda a_0 (a_0 - 1) \overline{z_0}.$$

Thus for a given nonzero  $\lambda$ , Eq. (65) holds in the case of  $n = 1$  for the above chosen Rota-Baxter algebra  $(\text{III}(\mathbf{k})/I_1, \overline{P_k})$  and  $f, g \in \text{III}(\mathbf{k})/I_1$  if and only if  $a_0 = 0$  or  $a_0 = 1$ , i.e.,  $\omega = xy$  or  $\omega = xy - 1$ . Next consider  $\omega = xy - (b_0y + yx)$ . Then applying Eq. (32) gives

$$(68) \quad \widehat{P}_1^\omega(f) = b_0 P(f_0) + P(f_1) \quad \text{for all } f \in R^\mathbb{N}.$$

We take  $(R, P) := (\text{III}(\mathbf{k})/I_2, \overline{P_k})$ , and  $f := (f_\ell), g := (g_k) \in (\text{III}(\mathbf{k})/I_2)^\mathbb{N}$  with  $f_\ell := \delta_{\ell,1} \overline{z_0}, g_k := \delta_{k,0} \overline{z_0}$ . Then we obtain

$$(69) \quad \widehat{P}_0^\omega(f) = \overline{0}, \quad \widehat{P}_0^\omega(g) = \overline{z_1}, \quad \widehat{P}_1^\omega(f) = P(f_1) = \overline{z_1}, \quad \widehat{P}_1^\omega(g) = b_0 P(g_0) = b_0 \overline{z_1}.$$

Thus

$$\begin{aligned} \sum_{k=0}^1 \sum_{j=0}^{1-k} \binom{1}{k} \binom{1-k}{j} \lambda^k \widehat{P}_{1-j}^\omega(f) \widehat{P}_{k+j}^\omega(g) &= \widehat{P}_0^\omega(f) \widehat{P}_1^\omega(g) + \widehat{P}_1^\omega(f) \widehat{P}_0^\omega(g) + \lambda \widehat{P}_1^\omega(f) \widehat{P}_1^\omega(g) \\ &= \overline{z_1}^2 + \lambda b_0 \overline{z_1}^2 = \lambda \overline{z_1} + \lambda^2 b_0 \overline{z_1} \end{aligned}$$



and

$$\begin{aligned}
& \widehat{P}_1^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g)) + \lambda\widehat{P}_1^\omega(fg) \\
&= \widehat{P}_1^\omega(\widehat{P}^\omega(f)) + \widehat{P}_1^\omega(f\widehat{P}^\omega(g)) + \lambda\widehat{P}_1^\omega(f) \quad (\text{since } g \text{ is the identity element}) \\
&= b_0P(\widehat{P}_0^\omega(f)) + P(\widehat{P}_1^\omega(f)) + b_0P((f\widehat{P}^\omega(g))_0) + P((f\widehat{P}^\omega(g))_1) \\
&\quad + \lambda(b_0P(f_0) + P(f_1)) \quad (\text{by Eq. (68)}) \\
&= b_0P(\widehat{P}_0^\omega(f)) + P(\widehat{P}_1^\omega(f)) + b_0P(f_0\widehat{P}_0^\omega(g)) + P(f_1\widehat{P}_0^\omega(g) + f_0\widehat{P}_1^\omega(g) + \lambda f_1\widehat{P}_1^\omega(g)) \\
&\quad + \lambda(b_0P(f_0) + P(f_1)) \quad (\text{by Eq. (6)}) \\
&= \lambda\overline{z_1} \quad (\text{by Eq. (69)}).
\end{aligned}$$

Thus for a nonzero  $\lambda \in \mathbf{k}$ , Eq. (65) holds in the case of  $n = 1$  for the above chosen Rota-Baxter algebra and  $f, g$  if and only if  $\lambda\overline{z_1} + \lambda^2 b_0\overline{z_1} - \lambda\overline{z_1} = \lambda^2 b_0\overline{z_1} = \overline{0}$ , which holds if and only if  $b_0 = 0$ . Thus  $\omega = xy - yx$ .

Therefore,  $\omega$  must be in  $\Omega_{\mathbf{k}} = \{xy, xy - 1, xy - yx\}$ .

(iib)  $\implies$  (iia). For  $\omega = xy - 1$ , the coextension  $\widehat{P}^\omega$  of  $P$  to  $(R^\mathbb{N}, \partial_R)$  is again a Rota-Baxter operator of weight  $\lambda$  by [26, Proposition 3.8]. When  $\omega = xy$  or  $\omega = xy - yx$ , we need to show that the coextension  $\widehat{P}^\omega$  of  $P$  to  $(R^\mathbb{N}, \partial_R)$  satisfies Eq. (65).

For  $\omega = xy$ , applying Eq. (32), we have

$$(70) \quad \widehat{P}_n^\omega(f) = 0 \quad \text{for all } n \in \mathbb{N}_+, f \in R^\mathbb{N}.$$

By Proposition 2.8.(i), Eq. (65) holds for  $n = 0$ . If  $n \in \mathbb{N}_+$ , then the maximum of the subscripts  $m$  of the expressions  $\widehat{P}_m^\omega$  appearing in each term of the left side of Eq. (65) is strictly larger than 0 and then by Eq. (70), each term in the left side of Eq. (65) is 0. Thus Eq. (65) holds for all  $n \in \mathbb{N}_+$ .

For  $\omega = xy - yx$ , by Eq. (32), the coextension  $\widehat{P}^\omega$  of  $P$  is given by

$$(71) \quad \widehat{P}_n^\omega(f) = \widehat{P}_{n-1}^\omega(\partial_R f) = \widehat{P}_0^\omega(\partial_R^n f) = P(f_n) \quad \text{for all } n \in \mathbb{N}, f \in R^\mathbb{N}.$$

Furthermore, for all  $f, g \in R^\mathbb{N}$ , and  $n \in \mathbb{N}$ ,

$$\begin{aligned}
& \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k \widehat{P}_{n-j}^\omega(f) \widehat{P}_{k+j}^\omega(g) \\
&= \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k P(f_{n-j}) P(g_{k+j}) \quad (\text{by Eq. (71)}) \\
&= \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k (P(P(f_{n-j})g_{k+j}) + P(f_{n-j}P(g_{k+j})) + \lambda P(f_{n-j}g_{k+j})) \quad (\text{by Eq. (3)}) \\
&= \sum_{k=0}^n \sum_{j=0}^{n-k} \binom{n}{k} \binom{n-k}{j} \lambda^k (P(\widehat{P}_{n-j}^\omega(f)g_{k+j}) + P(f_{n-j}\widehat{P}_{k+j}^\omega(g)) + \lambda P(f_{n-j}g_{k+j})) \quad (\text{by Eq. (71)}) \\
&= P((\widehat{P}^\omega(f)g)_n) + P((f\widehat{P}^\omega(g))_n) + \lambda P((fg)_n) \quad (\text{by Eq. (6)}) \\
&= \widehat{P}_n^\omega(\widehat{P}^\omega(f)g) + \widehat{P}_n^\omega(f\widehat{P}^\omega(g)) + \lambda\widehat{P}_n^\omega(fg). \quad (\text{by Eq. (71)})
\end{aligned}$$

Then Eq. (65) holds.

Now we have completed the proof of Theorem 4.1.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NJ 07102, USA  
 E-mail address: [liguo@newark.rutgers.edu](mailto:liguo@newark.rutgers.edu)

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, RUTGERS UNIVERSITY, NEWARK, NJ 07102, USA  
 E-mail address: [keigher@newark.rutgers.edu](mailto:keigher@newark.rutgers.edu)

DEPARTMENT OF MATHEMATICS, LANZHOU UNIVERSITY, LANZHOU, GANSU, 730000, CHINA  
 E-mail address: [2663067567@qq.com](mailto:2663067567@qq.com)